ANOSOV DIFFEOMORPHISMS WITH NON-TRIVIAL
RUUELLE SPECTRUM

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Abstract. We show that generic real-analytic perturbations of linear hyperbolic toral automorphisms are Anosov maps with non trivial Ruelle spectrum.

1. Statement of Result

The purpose of this short note is to provide a proof of the seemingly obvious fact that non linear Anosov diffeomorphisms of the torus have generically a non-trivial Ruelle spectrum. Let $T^d = \mathbb{R}^d / \mathbb{Z}^d$ be the flat torus, and pick $A \in \text{SL}_d(\mathbb{Z})$ a hyperbolic matrix. The linear map $A$ induces an Anosov map of the torus $T^d$, again denoted by $A : T^d \to T^d$. In this note, we will consider perturbations $A_\epsilon : T^d \to T^d$ of $A$ defined for small $\epsilon$ by

$$A_\epsilon(x) = Ax + \epsilon \Psi(x) \mod \mathbb{Z}^d,$$

where $\Psi : T^d \to \mathbb{R}^d$ is a real-analytic map. Here we are interested in the spectral properties of the composition operator

$$T_\epsilon(f) := f \circ A_\epsilon,$$

on a suitable Banach space.

By $\mathcal{B}_\rho(T^d)$ we denote the space of functions $\mathbb{R}^d \to \mathbb{C}$ which are $\mathbb{Z}^d$-periodic and have a holomorphic extension to

$$\mathcal{U}_\rho := \{ (z_1, \ldots, z_d) \in \mathbb{C}^d : \max_{1 \leq j \leq d} |\text{Im}(z_j)| < \rho \} \supset \mathbb{R}^d,$$

and we endow $\mathcal{B}_\rho(T^d)$ with the obvious norm

$$\|f\|_\rho := \sup_{z \in \mathcal{U}_\rho} |f(z)|.$$

Similarly, we set $\mathcal{B}_\rho(T^d, \mathbb{R})$ to be the closed subspace of $\mathcal{B}_\rho(T^d)$ given by functions which are $\mathbb{R}$-valued on $\mathbb{R}^d$. From the work of Faure-Roy [5] (at least for $d = 2$) we know that the following fact holds.

**Theorem 1.1.** Assume that $\Psi \in \mathcal{B}_\rho(T^d, \mathbb{R})^d$ for some $\rho > 0$. Then there exist $\overline{\rho}(\Psi) > 0$, a Hilbert space $\mathcal{H}_\overline{\rho}$ with

$$\mathcal{B}_\rho(T^d) \subset \mathcal{H}_\overline{\rho} \subset \mathcal{B}_\overline{\rho}(T^d),$$

such that for all $\epsilon > 0$ small enough, $T_\epsilon$ acts on $\mathcal{H}_\overline{\rho}$ as a compact trace class operator.

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Notice that this Hilbert space (which can be constructed using a Fourier basis) do contains hyperfunctions, i.e. analytic functionals. We shall call this spectrum the Ruelle spectrum of $T_\epsilon$. It is well known that if $\epsilon = 0$ this spectrum is exactly \{0\} $\cup$ \{1\}, which reflects the fact that linear Anosov maps have a superexponential rate of mixing, see for example [2], chapter 2.

In this note, we show the following.

**Theorem 1.2.** Fix $\rho > 0$. There exists an open and dense subset $\mathcal{G}$ of $\mathcal{B}_\rho (\mathbb{T}^d, \mathbb{R})^d$ such that for all $\Psi \in \mathcal{G}$, for all $\epsilon > 0$ small enough, $A_\epsilon : \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho$ has at least one non trivial eigenvalue $\lambda_\epsilon \notin \{0,1\}$.

2. Proof, semi-detailed

Let us prove Theorem 1.2. First notice that by structural stability, $A_\epsilon : \mathbb{T}^d \rightarrow \mathbb{T}^d$ has (for $\epsilon$ small), the same number of fixed points as $A_0$, which is exactly given by the integer $N_A := \mid \det(A - I) \mid$. Let us denote these fixed points by $x_j$, $j = 1, \ldots, N_A$. Each $x_j$ is solution of an equation

$$Ax_j - x_j = k_j, \quad k_j \in \mathbb{Z}^d.$$ 

A simple continuity argument shows that for all $\epsilon$ small enough, the fixed points $x_j(\epsilon)$ of $A_\epsilon$ are solutions of

$$Ax_j(\epsilon) + \epsilon \Psi(x_j(\epsilon)) - x_j(\epsilon) = k_j.$$ 

Because $\det(A - I) \neq 0$, the implicit function theorem (and also structural stability) implies easily that each $x_j(\epsilon)$ is a smooth function of $\epsilon \geq 0$ with an asymptotic expansion of the type

$$x_j(\epsilon) = x_j + \epsilon b_j + O(\epsilon^2),$$

for some $b_j \in \mathbb{R}^d$. By a result of Faure-Roy [5], we know that the trace of $T_\epsilon$ is given for all $\epsilon \geq 0$ small by

$$\text{Tr}(T_\epsilon) = \sum_{j=1}^{N_A} \frac{1}{\mid \det(I - D_{x_j(\epsilon)}A_\epsilon) \mid}.$$ 

For each $j = 1, \ldots, N_A$, we write

$$\mid \det(I - D_{x_j(\epsilon)}A_\epsilon) \mid = N_A \mid \det \left( I - \epsilon (I - A)^{-1} D_{x_j(\epsilon)} \Psi \right) \mid$$ 

$$= N_A \left( 1 - \epsilon \text{Tr} \left( (I - A)^{-1} D_{x_j} \Psi \right) + O(\epsilon^2) \right).$$

We therefore obtain

$$\text{Tr}(T_\epsilon) = 1 + \frac{1}{N_A} \sum_{j=1}^{N_A} \text{Tr} \left( (I - A)^{-1} D_{x_j} \Psi \right) + O(\epsilon^2).$$

Differentiation being continuous on $\mathcal{B}_\rho (\mathbb{T}^d)$ (this is a holomorphic function space), it is clear that the linear functional $L_A : \mathcal{B}_\rho (\mathbb{T}^d, \mathbb{R})^d \rightarrow \mathbb{R}$ given by

$$L_A(\Phi) = \frac{1}{N_A} \sum_{j=1}^{N_A} \text{Tr} \left( (I - A)^{-1} D_{x_j} \Phi \right)$$

is also bounded. We now need to show that \( \mathcal{L}_A \) is non-identically vanishing (on \( B_\rho(T^d) \)) which will follow from a standard approximation argument. First we remark that since trigonometric polynomials are \( C^1 \)-dense in \( C^\infty(T^d) \), it is enough to do it for \( C^\infty \) maps \( T^d \to \mathbb{R}^d \). Let \( \tilde{x}_j \) denote lifts of the fixed points \( x_j \) to the universal cover \( \mathbb{R}^d \). Pick \( \varphi_0 \in C^\infty_0(\mathbb{R}^d) \) such that \( \varphi_0 \equiv 1 \) on a neighbourhood of \( \tilde{x}_1 \) while \( \varphi_0 \equiv 0 \) on a neighbourhood of \( \tilde{x}_j \) for all \( j \neq 1 \). Set \( L := ( (I - A)^{-1})^T \), where \( M^T \) denotes the transpose of \( M \). Consider now the map \( \Psi : \mathbb{R}^d \to \mathbb{R}^d \) defined by \( \Psi(x) = \varphi_0(x)L(x) \). By averaging over \( \mathbb{Z}^d \), we can obtain now a map defined unambiguously on the torus

\[
\tilde{\Psi}(x) := \sum_{n \in \mathbb{Z}^d} \Psi(x - n).
\]

It is now a simple task to check that \( D_{x_j} \tilde{\Psi} = L \) while \( D_{x_j} \tilde{\Psi} = 0 \) for all \( j \neq 1 \), so that

\[
\mathcal{L}_A(\tilde{\Psi}) = \frac{1}{N_A} \text{Tr}((I - A)^{-1}L) \neq 0.
\]

As a consequence, \( \mathcal{G} = \mathcal{L}_A^{-1}(\mathbb{R} \setminus \{0\}) \) is an open and dense set. The claim is proved since for all \( \Psi \in \mathcal{G} \) we have

\[
\text{Tr}(T_\epsilon) = 1 + \epsilon L_A(\Psi) + O(\epsilon^2)
\]

with \( L_A(\Psi) \neq 0 \). □

Using the technology of [3, 4], it may be possible to extend this result for \( C^\infty \) perturbations. One might try to use the flat trace for high iterates of \( T_\epsilon \) to deal with the error term relating the flat trace to the discrete eigenvalues.

**Final Comment.** This sketchy argument has been made into a fully detailed proof by A. Adam recently in [1] where he also investigates the case of volume preserving perturbations. It would be interesting to address the problem in a more global manner i.e. by considering perturbations of general analytic Anosov maps, or Gevrey maps.

**References**


