Equidistribution of Eisenstein Series for Convex Co-compact Hyperbolic Manifolds

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The quantum ergodic theorem, due to Schnirelman [4], Colin de Verdière [1] and Zelditch [5], says that on any compact Riemannian manifold $X$ whose geodesic flow is ergodic, one can find a full density sequence $\lambda_j \to +\infty$ of eigenvalues of the Laplacian $\Delta_X$ such that the corresponding normalized eigenfunctions $\psi_j$ are equidistributed i.e. for all $f \in L^2(X)$, we have

$$\lim_{j \to +\infty} \int_X f(z) |\psi_j(z)|^2 dv(z) = \int_X f(z) dv(z).$$

where $dv$ is the normalized volume measure. For non-compact manifolds, there can be continuous spectrum and the quantum ergodic theorem does not really make sense in general. However, for hyperbolic surfaces of finite volume and in particular arithmetic cases, Zelditch [6], Luo-Sarnak [3] prove a related statement involving the generalized eigenfunctions (also known as Eisenstein series). Let us recall their results. Let $X = \Gamma \backslash \mathbb{H}^2$ be a finite area surface where $\Gamma$ is a non co-compact co-finite Fuchsian group. The non compact ends of $X$ are cusps related to fixed points $c_j$ in $\partial \mathbb{H}^2$ of parabolic elements in $\Gamma$. The spectrum of the Laplacian $\Delta_X$ has a discrete part which corresponds to $L^2(X)$-eigenfunctions and may be infinite and the absolutely continuous part $[1/4, +\infty)$ which is parametrized ($t \in \mathbb{R}$) by the finite set of Eisenstein series $E_X(1/2 + it; z, j)$ related to each cusp $c_j$. The Eisenstein series $E_X(1/2 + it; z, j)$ are smooth non-$L^2(X)$ eigenfunctions.

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\[ \Delta_X E_X(1/2 + it; z, j) = (1/4 + t^2) E_X(1/2 + it; z, j). \]

For all \( t \in \mathbb{R} \), define the density \( \mu_t \) by

\[
\int_X a(z) d\mu_t(z) := \sum_j \int_X a(z)|E_X(1/2 + it; z, j)|^2 \, dv(z),
\]

where \( a \in C_0^\infty(X) \). In the case with only finitely many eigenvalues then Zelditch’s equidistribution result is as follows: for \( a \in C_0^\infty(X) \),

\[
\frac{1}{s(T)} \int_{-T}^{T} \left| \int_X a(z) - \partial_t s(t) \int_X a \, dv \right| \, dt \to 0 \quad \text{as} \quad T \to \infty
\]

where \( s(t) \) is the scattering phase appearing as a sort of regularization of Eisenstein series due to the fact that the Weyl law involves the continuous spectrum. On the other hand, for the modular surface \( X = \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2 \), Luo and Sarnak [3] showed that as \( t \to +\infty \),

\[
\int_X a d\mu_t = \frac{48}{\pi} \log(t) \int_X a \, dv + o(\log(t)),
\]

which is a much stronger statement obtained via sharp estimates on certain \( L \)-functions.

We report here some recent result of [2], where we studied the case of \textit{infinite volume hyperbolic manifolds} without cusps, more precisely convex co-compact quotients \( X = \Gamma \backslash \mathbb{H}^{n+1} \) of the hyperbolic space. A discrete group of orientation preserving isometries of \( \mathbb{H}^{n+1} \) is said to be convex co-compact if it admits a polygonal, finite sided fundamental domain whose closure does not intersect the limit set of \( \Gamma \). The limit set \( \Lambda_\Gamma \) and the set of discontinuity \( \Omega_\Gamma \) are defined by

\[
\Lambda_\Gamma := \overline{\Gamma \cdot o} \cap S^n, \quad \Omega_\Gamma := S^n \setminus \Lambda_\Gamma,
\]

where \( o \in \mathbb{H}^{n+1} \) is any point in \( \mathbb{H}^{n+1} \). The quotient space \( X = \Gamma \backslash \mathbb{H}^{n+1} \) has ‘funnel type’ ends and is the interior of a compact manifold with boundary \( X \) := \( \Gamma \backslash (\mathbb{H}^{n+1} \cup \Omega_\Gamma) \), the action of \( \Gamma \) on \( (\mathbb{H}^{n+1} \cup \Omega_\Gamma) \) being free and totally discontinuous. By a result of Patterson and Sullivan, the Hausdorff dimension of \( \Lambda_\Gamma \)

\[
\delta_\Gamma := \dim_{\text{Haus}}(\Lambda_\Gamma)
\]

is also the exponent of convergence of the Poincaré series, i.e. for all \( m, m' \in \mathbb{H}^{n+1} \) and \( s > 0 \),

\[
\sum_{\gamma \in \Gamma} e^{-s d(\gamma m, m')} < \infty \iff s > \delta_\Gamma,
\]

where \( d(m, m') \) denotes the hyperbolic distance.
In that case the spectrum of $\Delta_X$ consists of the absolutely continuous spectrum $[n^2/4, +\infty)$ and a (possibly empty) finite set of eigenvalues in $(0, n^2/4)$. The Eisenstein functions are defined using the ball model of $\mathbb{H}^{n+1}$ to be the automorphic functions of $m \in \mathbb{H}^{n+1}$ given by

$$E_X(s; m, \xi) = \sum_{\gamma \in \Gamma} \left( \frac{1 - |\gamma m|^2}{4|\gamma m - \xi|^2} \right)^s, \quad \xi \in \Omega_\Gamma,$$

which are absolutely convergent for $\text{Re}(s) > \delta_\Gamma$ and extend meromorphically to $s \in \mathbb{C}$. The Eisenstein series are non-$L^2(X)$ eigenfunctions of the Laplacian with eigenvalue $s(n - s)$ on $\text{Re}(s) = n/2$. We show the following

**Theorem 1.** Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact quotient with $\delta_\Gamma < n/2$. Let $a \in C^\infty_0(X)$ and let $E_X(s; \cdot, \xi)$ be an Eisenstein series as above with a given point $\xi \in \partial X$ at infinity. Then we have as $t \to +\infty$,

$$\int_X a(m) \left| E_X \left( \frac{n}{2} + it; m, \xi \right) \right|^2 d\nu(m) = \int_X a(m) E_X(n; m, \xi) d\nu(m) + O(t^{2\delta_\Gamma - n}).$$

The limit measure on $X$ is given by the harmonic density $E_X(n; m, \xi)$ whose boundary limit is the Dirac mass at $\xi \in \partial X$. A microlocal extension of this theorem is also proved. We first need to introduce some adequate notations. Fix any $\xi \in \partial X$. Let $L_{\gamma \xi}$ defined by

$$L_{\gamma \xi} := \bigcup_{\gamma \in \Gamma} L_{\gamma \xi} \subset S^* X,$$

where $L_{\gamma \xi}$ are stable Lagrangian submanifolds of the unit cotangent bundle $S^* X$: the Lagrangian manifold $L_{\gamma \xi}$ is defined to be the projection on $\Gamma \backslash S^* \mathbb{H}^{n+1}$ of

$$\{(m, v_{\gamma \xi}(m)) \in S^* \mathbb{H}^{n+1}; m \in \mathbb{H}^{n+1}\},$$

where $v_{\gamma \xi}(m)$ is the unit (co)vector tangent to the geodesic starting at $m$ and pointing toward $\gamma \xi \in S^n$. The set $L_{\gamma \xi}$ “fibers” over $X$, and the fiber over a point $m \in X$ corresponds to the closure of the set of directions $v \in S^* X$ such that the geodesic starting at $m$ with directions $v$ converges to $\xi \in \partial X$ as $t \to +\infty$. Since the closure of the orbit $\Gamma.\xi$ satisfies $\Gamma.\xi \supset \Lambda_\Gamma$, $L_{\gamma \xi}$ contains the forward trapped set

$$T_+ := \{(m, v) \in S^* X : g_t(m, v) \text{ remains bounded as } t \to +\infty\},$$

where $g_t : S^* X \to S^* X$ is the geodesic flow. The Hausdorff dimension of $L_{\gamma \xi}$ is $n + \delta_\Gamma + 1$ and satisfies $n + 1 < \delta_\Gamma + n + 1 < 2n + 1$ if $\Gamma$ is non elementary.

Our phase-space statement is the following

**Theorem 2.** Let $A$ be a compactly supported $0$-th order pseudodifferential operator with principal symbol $a \in C^\infty_0(X, T^* X)$, then as $t \to +\infty$
\( \left( A E_X \left( \frac{n}{2} + it; \cdot, \xi \right), E_X \left( \frac{n}{2} + it; \cdot, \xi \right) \right)_{L^2_1(X)} = \int_{S^* X} a \, d\mu_\xi + O(t^{-\min(1,n-2\delta)}) \)

where \( \mu_\xi \) is a \( g_t \)-invariant measure supported on the fractal subset \( L_\xi \subset S^* X \).

Notice that the fractal behaviour of the semi-classical limit \( \mu_\xi \) can only be observed at the microlocal level. By averaging over the boundary with respect to the volume measure induced by \( S^n \) on \( \Omega \Gamma \), we obtain as \( t \to +\infty \)

\[
\int_{\Omega} \int_X a(m) \left| E_X \left( \frac{n}{2} + it; m, \xi \right) \right|^2 \, dv(m) \, d\xi = \vol(S^n) \int_X a(m) \, dv(m) + O(t^{2\delta - n})
\]

and

\[
\int_{\Omega} \left( A E_X \left( \frac{n}{2} + it; \cdot, \xi \right), E_X \left( \frac{n}{2} + it; \cdot, \xi \right) \right)_{L^2_1(X)} \, d\xi = \int_{S^* X} a \, d\mu + O(t^{-\min(n-2\delta,1)})
\]

where \( \mu \) denotes the Liouville measure. This is the perfect analog of the previously known results for the modular surface (actually with a remainder in our case).

References