

# ON THE NODAL LINES OF EISENSTEIN SERIES ON SCHOTTKY SURFACES

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ABSTRACT. On convex co-compact hyperbolic surfaces  $X = \Gamma \backslash \mathbb{H}^2$ , we investigate the behavior of nodal curves of real valued Eisenstein series  $F_\lambda(z, \xi)$ , where  $\lambda$  is the spectral parameter,  $\xi$  the direction at infinity. Eisenstein series are (non- $L^2$ ) eigenfunctions of the Laplacian  $\Delta_X$  satisfying  $\Delta_X F_\lambda = (\frac{1}{4} + \lambda^2)F_\lambda$ . As  $\lambda$  goes to infinity (the high energy limit), we show that, for generic  $\xi$ , the number of intersections of nodal lines with any compact segment of geodesic grows like  $\lambda$ , up to multiplicative constants. Applications to the number of nodal domains inside the convex core of the surface are then derived.

## 1. INTRODUCTION

Let  $\mathbb{H}^2$  be the hyperbolic plane endowed with the usual metric of constant negative curvature  $-1$ . Assume that  $\Gamma$  is a convex co-compact group of isometries, i.e. a Schottky group with no parabolic elements, and denote by  $X = \Gamma \backslash \mathbb{H}^2$  the quotient surface. Such a surface has infinite area and the ends are hyperbolic funnels. Let  $\Delta_X$  denote the *positive* hyperbolic Laplacian on  $X$ . Its  $L^2$ -spectrum has been described completely by Lax and Phillips in [14]. The half line  $[1/4, +\infty)$  is the continuous spectrum and it contains no embedded eigenvalues. Let  $\delta(\Gamma)$  be the *Hausdorff* dimension of the limit set  $\Lambda(\Gamma)$  of  $\Gamma$ . The limit set  $\Lambda(\Gamma)$  is defined as the set of accumulation points in  $\partial\mathbb{H}^2$  of the orbit of any point  $z \in \mathbb{H}^2$  under the action of  $\Gamma$ :

$$\Lambda(\Gamma) := \overline{\Gamma \cdot z} \cap \partial\mathbb{H}^2.$$

The rest of the spectrum (point spectrum) is empty if  $\delta \leq \frac{1}{2}$ , on the other hand it is finite and starting at  $\delta(1 - \delta)$  if  $\delta > \frac{1}{2}$ . The fact that the bottom of the spectrum is related to the dimension  $\delta$  is due to Patterson [19]. One way to parametrize the continuous spectrum is through the so-called Eisenstein Series. Before we can give a formal definition of Eisenstein series, let us recall that under the above assumptions, when  $\Gamma$  is non-elementary (i.e.  $X$  is not a hyperbolic cylinder),  $X$  can be decomposed as

$$X = X_0 \cup \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{n_f},$$

where  $X_0$  is a compact surface with geodesic boundary and  $\mathcal{F}_1, \dots, \mathcal{F}_{n_f}$  are the funnels. Each funnel  $\mathcal{F}_j$  is isometric to the cylinder

$$(0, 2]_\rho \times (\mathbb{R}/\ell_j\mathbb{Z})_\theta,$$

endowed with the conformally compact metric

$$ds^2 = \frac{d\rho^2 + (1 + \rho^2/4)d\theta^2}{\rho^2},$$

where  $\rho = 0$  corresponds to infinity and  $\ell_j$  is the length of the geodesic boundary at  $\rho = 2$ . Let  $R_X(s; z, w)$  denote the Schwarz kernel of the resolvent  $(\Delta_X - s(1 - s))^{-1}$  which by Mazzeo-Melrose [16] has a meromorphic continuation (in  $s$ ) to the whole

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complex plane. Then if  $s$  is not a pole, the limit (using the above coordinates in the funnel)

$$E_s(z, \xi) := \lim_{\rho \rightarrow 0} \rho^{-s} R_X(s; z, (\rho, \xi))$$

exists and defines an *eigenfunction* of the Laplacian  $\Delta_X E_s(z, \xi) = s(1-s)E(s; z, \xi)$ , parametrized by a point  $\xi$  at infinity, called *Eisenstein series*. In particular if  $s = 1/2 + i\lambda$ , we have

$$\Delta_X E_s(z, \xi) = \left( \frac{1}{4} + \lambda^2 \right) E_s(z, \xi).$$

These Eisenstein series, like their analog in the finite volume case, provide an explicit spectral resolution of the Laplacian [1]:

$$2\lambda d\Pi_X(\lambda, z, z') = \frac{|C(1/2 + i\lambda)|^2}{2\pi} \sum_{j=1}^{n_f} \int_0^{\ell_j} E_{1/2+i\lambda}(z, \xi) E_{1/2-i\lambda}(z', \xi) d\xi,$$

where

$$C(s) = \frac{2^{-s} \Gamma(s)}{\sqrt{\pi} \Gamma(s - 1/2)}.$$

In this paper we want to investigate the zeros sets of *high energy Eisenstein series*, so we consider real valued Eisenstein functions i.e. we take real parts <sup>1</sup>(which are again eigenfunctions) and set for all  $z \in X$  and  $\xi$  a direction at infinity,

$$F_\lambda(z, \xi) := \operatorname{Re} \left( E_{\frac{1}{2}+i\lambda}(z, \xi) \right).$$

It is a natural question to investigate the shape and behaviour of the zeros sets (also called nodal lines in dimension 2) of  $F_\lambda(z, \xi)$  as the frequency  $\lambda$  goes to infinity. For genuine  $L^2$ -eigenfunctions on compact manifolds, there is a tremendous amount of work in that direction, and we refer the reader to the recent survey [23]. However, in the non-compact case and infinite volume case, this seems, to our knowledge, to be the very first related work. Numerical experiments show that nodal lines exhibit a mixed behaviour: horocyclic shape close to infinity while in the compact core they look more like a genuine high energy eigenfunction. We refer the reader to the Appendix, where we detail our numerical analysis and show several plots of nodal lines for a pair of symmetric pants.

In the case when  $\delta(\Gamma) < 1/2$ , then the lift to  $\mathbb{H}^2$  of  $F_\lambda(z, \xi)$  admits a convergent series expression (in the unit disc model),

$$(1) \quad F_\lambda(z, \xi) = \sum_{\gamma \in \Gamma} \left( \frac{1 - |\gamma z|^2}{|\gamma z - \xi|^2} \right)^{1/2} \cos \left( \lambda \log \left( \frac{1 - |\gamma z|^2}{|\gamma z - \xi|^2} \right) \right),$$

where  $z \in \mathbb{H}^2$  and  $\xi \in \partial\mathbb{H}^2$  belongs to the domain of discontinuity of  $\Gamma$  that is  $\partial\mathbb{H}^2 \setminus \Lambda(\Gamma)$ . See [8], Lemma 5, for a proof of that fact. For each term in this sum, the phase function has its level sets on Horocycles based at the point  $\gamma^{-1}\xi$  at infinity, so it is basically a superposition of *hyperbolic plane waves*.

Because  $F_\lambda$  is an eigenfunction of an elliptic operator with real analytic coefficients (the hyperbolic laplacian), it is automatically a real analytic function. The nodal sets  $\mathcal{N}_\lambda(\xi)$  are defined as usual by

$$\mathcal{N}_\lambda(\xi) := \{z \in X : F_\lambda(z, \xi) = 0\}.$$

These sets are real analytic curves (with possible isolated singular points) and therefore rectifiable. The nodal domains are the connected components of  $X \setminus \mathcal{N}_\lambda(\xi)$ . It is not difficult to see <sup>2</sup> that there are infinitely many nodal domains, and infinitely

<sup>1</sup>Results and proofs are the same for imaginary parts

<sup>2</sup>See Appendix 1 for more details

many non-compact nodal lines. We need therefore to restrict all counting problems to *suitable compact subsets of  $X$* .

Let  $\sigma$  denote the length measure induced on  $\mathcal{N}_\lambda(\xi)$ , then by translating almost verbatim the arguments of Donnelly-Feffermann [5] (which is a purely local proof), one obtains that for all compact  $K \subset X$  with non-empty interior, there exists  $C_K > 0$  such that as  $\lambda \rightarrow \infty$ ,

$$C_K^{-1}\lambda \leq \sigma(\mathcal{N}_\lambda(\xi) \cap K) \leq C_K\lambda.$$

In this paper we go beyond by proving the following result. Given a geodesic  $\mathcal{C}$ , we will define a notion of  $\xi$ -non symmetry ( $\xi$ -NS), see §3, which rules out cases where the geodesic  $\mathcal{C}$  is an axis of symmetry for certain geodesics related to  $\xi$ . The following holds.

**Theorem 1.1.** *Assume that  $\delta(\Gamma) < \frac{1}{2}$ . Let  $\mathcal{C}$  be a geodesic which satisfies  $\xi$ -NS. Then for all compact non empty segment  $\mathcal{C}_0 \subset \mathcal{C}$ , one can find a constant  $C$  such that as  $\lambda$  goes to infinity, we have*

$$C^{-1}\lambda \leq \#(\mathcal{N}_\lambda(\xi) \cap \mathcal{C}_0) \leq C\lambda.$$

The above statement is non-empty : for all geodesic  $\mathcal{C}$ ,  $\xi$ -NS is satisfied for almost all directions  $\xi$  at infinity, see §3. The upper bound is actually valid in greater generality for real analytic curves and generic  $\xi$ , see comments in §3. The constant  $C$  depends a priori on the curve  $\mathcal{C}_0$  itself and  $\xi$ .

We point out that several recent papers also focus on proving upper and lower bounds on the number of intersections of nodal lines with geodesics segments. On compact non positively curved surfaces with boundary, Jung and Zelditch [13], show that  $\#(\mathcal{N}_\lambda \cap \mathcal{C}_0)$  goes to infinity as  $\lambda$  goes to infinity, when  $\mathcal{C}$  is a boundary curve. On the other hand, a similar statement holds [12] on a negatively curved surface (without boundary) and when  $\mathcal{C}$  satisfies a non symmetry condition. On the modular surface  $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2$ , Jung [10] obtains effective lower bounds of the type

$$C^{-1}\lambda_k^{\frac{1}{8}-\epsilon} \leq \#(\mathcal{N}_{\lambda_k} \cap \mathcal{C}_0),$$

for the Maass-Hecke eigenfunctions (with discrete spectral parameter  $\lambda_k$  as in our case) for a large portion of  $\lambda_k$ 's and when  $\mathcal{C}$  is a vertical geodesic segment in the modular domain. In [7], Ghosh, Reznikov and Sarnak, assuming Lindelöf's hypothesis, obtained a related lower bound

$$C^{-1}\lambda^{\frac{1}{12}-\epsilon} \leq \#(\mathcal{N}_\lambda \cap \mathcal{C}_0),$$

for all  $\lambda_k$  large enough. Their Proof works also for Eisenstein series related to the cusp. See also [11] for related results on compact hyperbolic surfaces. On the Flat 2-torus, Bourgain and Rudnick [2] were able to show that for *non geodesic* curves,

$$\#(\mathcal{N}_\lambda \cap \mathcal{C}_0) \geq C\lambda^{1-\epsilon}.$$

It seems to us that Theorem 1.1 is the only optimal counting result so far. Of course our setup of infinite volume is helping us somehow, although there are some different technical difficulties to overcome. Theorem 1.1 is a consequence of the following (restriction) equidistribution result, which is of interest in itself.

**Theorem 1.2.** *Let  $\Gamma$  be a convex co-compact group with  $\delta(\Gamma) < \frac{1}{2}$ . Let  $\mathcal{C}_0$  be a finite length geodesic segment of a geodesic satisfying  $\xi$ -NS. Then for all  $\varphi \in C^1(\mathcal{C}_0)$ , we have*

$$\lim_{\lambda \rightarrow +\infty} \int_{\mathcal{C}_0} (F_\lambda(x, \xi))^2 \varphi(x) d\sigma(x) = \frac{1}{2} \int_{\mathcal{C}_0} E_1(x, \xi) \varphi(x) d\sigma(x),$$

where  $E_1(z, \xi)$  is the positive harmonic Eisenstein series at  $s = 1$ . More generally, the same statement holds for all real-analytic compact curve  $\mathcal{C}_0$ , for a generic choice of  $\xi$ .

This above theorem is a "restriction" version of the main equidistribution result of [8], and this is where the  $\xi$ -non symmetry assumption is required. Similar equidistribution restriction results are known on compact manifolds (so-called "QER" ) when the geodesic flow is ergodic, also under a non symmetry assumption, see for example Toth-Zelditch [21]. We also refer to the paper of Dyatlov-Zworski [6] for a semi-classical framework that generalizes the preceding results. See also Bourgain-Rudnick [2] for related results on the torus.

Most of the above mentioned works are motivated by the study of nodal domains. It is a notoriously challenging problem to count them and [13, 12, 7] provide the very first (deterministic) examples of eigenfunctions where one is actually able to show that the number of nodal domains goes to infinity at high frequency. As a corollary of theorem 1.1, we prove the following. Assume that  $\Gamma$  is non-elementary, and let  $X_0$  denote the convex core of  $X$  (the compact part with funnels removed) and let  $M_\xi(\lambda)$  be the number of (open) connected components of

$$\text{Int}(X_0) \setminus \mathcal{N}_\lambda(\xi).$$

**Corollary 1.3.** *Under the above hypotheses, for almost all  $\xi$ , there exists a constant  $C > 0$  such that for all  $\lambda \geq 1$ , we have*

$$M_\xi(\lambda) \leq C\lambda^2.$$

It is important to notice that the upper bound, which is the analog of Courant's nodal domain theorem, is *not obvious* in our case: eigenfunctions  $F_\lambda(z, \xi)$  do not satisfy any boundary condition on  $\partial X_0$ . It is tempting to believe that the upper bound is optimal, but we have no serious clue so far.

The plan of the paper is as follows. In §2 we recall some basic facts about hyperbolic planes waves and Eisenstein Series. In §3 we prove Theorem 1.2 and a result on the asymptotic average on a geodesic segment (Proposition 3.1). Theorem 1.2 will be used for both lower and upper bounds in the proof of Theorem 1.1, while Proposition 3.1 is critical for the lower bound, see §4 for details. We point out that while the lower bounds and the equidistribution result rely on elementary real analysis (stationary and non-stationary phase principles for oscillatory integrals), the upper bound requires some complex analysis. This problem is already present for compact manifolds where the upper bound of Donnelly-Feffermann has not yet been proved in the  $C^\infty$  category (Yau's conjecture). Because the methods we use here are fairly elementary and robust, we expect these set of results to be extendable to variable curvature cases, with a negative pressure condition, as long as some analyticity is available.

In the present paper, we have not addressed the problem of describing the nodal structure at infinity, and we just give a proof of the fact that there exist infinitely many non compact nodal lines in the vicinity of  $\xi$ . We believe that it is very likely that using a precise description of the resolvent asymptotics in the funnels, one could obtain a better picture of the nodal topology at infinity. It would also be relevant to try to prove a lower bound as in [13, 7] on the number of nodal domains when a symmetry leaves invariant both the surface and the eigenfunction  $F_\lambda(x, \xi)$ , the main obstacle being the fact that the  $\xi$ -non-symmetry condition is likely to be violated in that case. We conclude these remarks by pointing out that the work of Zelditch [24, 22] on complex zeros of eigenfunctions could very well find an infinite volume analog by studying zeros of complexifications of Eisenstein series along trapped and non-trapped geodesics. It would be interesting to see if

the limit distribution has a universal character (as for quantum ergodic sequence of eigenfunctions) or behaves more like toral eigenfunctions (integrable systems). This should be pursued elsewhere.

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## 2. BASIC ESTIMATES AND CONVERGENCE

In this section we gather various basic estimates that will be required later on. We start with some facts on Busemann functions that will be used frequently throughout the paper.

**2.1. Busemann functions.** We will mostly work with the unit disc model

$$\mathbb{H}^2 = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

endowed with the metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - |z|^2)^2}.$$

The hyperbolic distance between 0 and  $z$  is given by

$$(2) \quad d(0, z) = \log \left( \frac{1 + |z|}{1 - |z|} \right).$$

Given two points  $z, w \in \mathbb{H}^2$  and  $\xi \in \partial\mathbb{H}^2$ , i.e.  $|\xi| = 1$ , the *Busemann function*  $B_\xi(z, w)$  by

$$B_\xi(z, w) = \lim_{t \rightarrow +\infty} d(z, \xi_t) - d(w, \xi_t),$$

where  $t \mapsto \xi_t$  is converging to  $\xi$  as  $t \rightarrow +\infty$ . From that definition one deduces several standard properties of Busemann functions which can be checked easily.

- For all  $z, w, y$ , we have  $B_\xi(z, y) = B_\xi(z, w) + B_\xi(w, y)$ .
- For all  $z, w$ ,  $B_\xi(z, w) = -B_\xi(w, z)$ .
- For all isometry  $g$  of the hyperbolic plane, we have  $B_\xi(gz, gw) = B_{g^{-1}\xi}(z, w)$ .
- The formula holds:

$$B_\xi(z, w) = \log \left( \frac{(1 - |w|^2)|z - \xi|^2}{(1 - |z|^2)|w - \xi|^2} \right).$$

The level sets of  $z \mapsto B_\xi(z, w)$  are *Horocycles* based at  $\xi$ . The hyperbolic analog of monochromatic plane waves are functions of the form:

$$z \mapsto e^{i\lambda B_\xi(0, z)}.$$

It is shown in [8] that if  $\delta(\Gamma) < \frac{1}{2}$ , then generalized eigenfunctions  $E_{1/2+i\lambda}(z, \xi)$  which are a priori defined through analytic continuation admit the convergent series formula

$$E_{\frac{1}{2}+i\lambda}(z, \xi) = \sum_{\gamma \in \Gamma} e^{(\frac{1}{2}+i\lambda)B_\xi(0, \gamma z)}.$$

In particular, the formula for the (real) Eisenstein series becomes

$$F_\lambda(z, \xi) = \sum_{\gamma \in \Gamma} e^{\frac{1}{2}B_\xi(0, \gamma z)} \cos(\lambda B_\xi(0, \gamma z)).$$

We start by a simple estimate.

**Lemma 2.1.** *Assume that  $z \in K$ , where  $K$  is a compact subset of  $\mathbb{H}^2$ , and that  $\xi$  is not in the limit set of  $\Gamma$ . Then for all multi index  $\alpha = (\alpha_1, \dots, \alpha_N)$ , there exists a constant  $C(K, \alpha, \xi)$  such that*

$$|\partial_\alpha \left( e^{\frac{1}{2}B_\xi(0, \gamma z)} \right)| \leq e^{-\frac{1}{2}d(0, \gamma 0)} C(K, \alpha),$$

where  $z = x_1 + ix_2$  and  $\partial_\alpha = \frac{\partial}{\partial x_{\alpha_1}} \dots \frac{\partial}{\partial x_{\alpha_N}}$ .

*Proof.* We only compute the first derivatives, the rest follows by an easy induction. Writing by the cocycle property

$$e^{\frac{1}{2}B_\xi(0, \gamma z)} = e^{\frac{1}{2}(B_\xi(0, \gamma 0) + B_{\gamma^{-1}\xi}(0, z))},$$

we have for  $j = 1, 2$ :

$$\partial_j \left( e^{\frac{1}{2}B_\xi(0, \gamma z)} \right) = \left( -\frac{2x_j}{1 - |z|^2} - \frac{2((\gamma^{-1}\xi)_j - x_j)}{|\gamma^{-1}\xi - z|^2} \right) e^{\frac{1}{2}B_\xi(0, \gamma 0)}.$$

Now remark that by formula (2) we have

$$B_\xi(0, \gamma 0) = \log \left( \frac{1 - |\gamma 0|^2}{|\gamma 0 - \xi|^2} \right) = -d(0, \gamma 0) + 2 \log \left( \frac{1 + |\gamma 0|}{|\gamma 0 - \xi|} \right).$$

Because  $\xi$  is not in the limit set of  $\Gamma$ , the distance  $|\gamma 0 - \xi|$  is uniformly bounded from below, so there exists a constant  $C_1 > 0$  such that

$$B_\xi(0, \gamma 0) \leq -d(0, \gamma 0) + C_1.$$

Since  $z$  is confined to a compact set  $K$  and  $\gamma^{-1}\xi$  remains on the unit circle, the proof is done.  $\square$

This simple estimate implies that if  $\delta(\Gamma) < \frac{1}{2}$ , then the series defining  $F_\lambda(z, \xi)$  (and the derivatives) are uniformly convergent on every compact subset of  $\mathbb{H}^2$ . Indeed we recall that *Poincaré Series*

$$P_\Gamma(s) = \sum_{\gamma \in \Gamma} e^{-sd(0, \gamma 0)}$$

are convergent for all  $s > \delta(\Gamma)$ , see [19]. Therefore,  $F_\lambda(z, \xi)$  is a  $C^\infty$  function, which is not surprising. From this elementary estimate, we have readily the following consequence which is worth highlighting.

**Proposition 2.2.** *For every compact set  $K \subset \mathbb{H}^2$ , there exist  $C_K > 0$  independent of  $\lambda \gg 1$  such that*

$$(3) \quad \|F_\lambda\|_{L^\infty(K)} \leq C_K.$$

Notice that in the compact or finite volume case,  $L^\infty$  norms of high energy eigenfunctions are usually not expected to be bounded: For example  $L^2$ -normalized eigenfunctions on arithmetic hyperbolic surfaces are not  $L^\infty$  bounded, see [15].

### 3. RESTRICTION THEOREMS

In this section we prove the main equidistribution theorem for restriction to geodesics (and more) stated in the introduction. Most of the results will rest on repeated applications of stationary and non-stationary phase formulas.

**3.1. Asymptotic average on a geodesic segment.** Geodesics in the disc model will be parametrized in the following way. Let  $g$  be a Möbius map of the unit disc, i.e. an homographic map of the form

$$g(z) = \frac{az + b}{bz + \bar{a}},$$

with  $a, b \in \mathbb{C}$  so that  $|a|^2 - |b|^2 = 1$ . Then the image of

$$g : (-1, +1) \rightarrow \mathbb{H}^2$$

is a geodesic. We denote it by  $\mathcal{C}_g$ . Conversely all geodesics of the disc can be viewed that way: given a geodesic  $\mathcal{C}$  and a point  $z_0 \in \mathcal{C}$ , there exists a Möbius transform  $g$  such that  $g(0) = z_0$  and  $g((-1, +1)) = \mathcal{C}$ . What we first prove is the following.

**Proposition 3.1.** *Assume that  $\delta(\Gamma) < \frac{1}{2}$ . Let  $\mathcal{C}_{r_0}$  be a geodesic segment in  $\mathbb{H}^2$ , parametrized by  $g : [-r_0, +r_0] \rightarrow \mathbb{H}^2$ . Then there exists a non empty open interval  $J \subset [-r_0, +r_0]$  and  $C > 0$  such that as  $\lambda$  goes to infinity, we have*

$$\sup_{\alpha < \beta \in J} \left| \int_{\alpha}^{\beta} F_{\lambda}(g(r), \xi) dr \right| \leq \frac{C}{\lambda}.$$

*Proof.* Since  $\delta < \frac{1}{2}$ , using the representation of  $F_{\lambda}$  as a sum of convergent series we are left with estimating the sum

$$\sum_{\gamma \in \Gamma} \int_{\alpha}^{\beta} e^{(\frac{1}{2} + i\lambda)B_{\xi}(0, \gamma g(r))} dr,$$

where  $\alpha < \beta \in J \subset [-r_0, +r_0]$  and  $J$  has to be chosen. The choice of  $J$  will follow from a careful analysis of the stationary points of the phase  $B_{\xi}(0, \gamma g(z))$ . Writing

$$B_{\xi}(0, \gamma g(r)) = B_{\xi}(0, \gamma g(0)) + B_{g^{-1}\gamma^{-1}(\xi)}(0, r),$$

we deduce that

$$(4) \quad \frac{d}{dr}(B_{\xi}(0, \gamma g(r))) = 2 \frac{r^2 a_{\gamma} - 2r + a_{\gamma}}{(1 - r^2)((r - a_{\gamma})^2 + b_{\gamma}^2)},$$

where we have set  $a_{\gamma} = \operatorname{Re}(g^{-1}\gamma^{-1}(\xi))$ ,  $b_{\gamma} = \operatorname{Im}(g^{-1}\gamma^{-1}(\xi))$ . The critical points are then given by

$$r_{\gamma}^{\pm} = \frac{1}{a_{\gamma}} \left( 1 \pm \sqrt{1 - a_{\gamma}^2} \right)$$

if  $a_{\gamma} \neq 0$ , and 0 otherwise. Remark that only  $r_{\gamma}^{-}$  can be a critical point ( $r_{\gamma}^{+}$  is outside the disc) and we have if  $a_{\gamma} \neq 0$ ,

$$\begin{aligned} \left| \frac{d}{dr}(B_{\xi}(0, \gamma g(r))) \right| &\geq \frac{1}{2} |a_{\gamma}| |r - r_{\gamma}^{+}| |r - r_{\gamma}^{-}| \\ &\geq \frac{1}{2} |r - r_{\gamma}^{-}| (1 - r_0). \end{aligned}$$

More precisely, consider the continuous, injective map

$$F : [-1, +1] \rightarrow [-1, +1]$$

defined by

$$F(x) = \frac{x}{1 + \sqrt{1 - x^2}},$$

then  $F(a_{\gamma})$  is the unique possible critical point of the phase. In all cases, we have a lower bound for the derivative: for all  $r \in [-r_0, +r_0]$ ,

$$\left| \frac{d}{dr}(B_{\xi}(0, \gamma g(r))) \right| \geq C(r_0) |r - F(a_{\gamma})|,$$

where  $C(r_0)$  is uniform in  $\gamma$ . The goal is now to find a non empty interval that is uniformly away from all the critical points. Let us consider

$$K := \overline{\bigcup_{\gamma \in \Gamma} g^{-1} \circ \gamma^{-1}(\xi)} \subset \partial \mathbb{H}^2,$$

and set for all  $z \in \partial \mathbb{H}^2$ ,  $\tilde{F}(z) = F(\operatorname{Re}(z))$ . Then define

$$\mathcal{B} := \tilde{F}(K) \cap [-r_0, +r_0],$$

then  $\mathcal{B}$  is a compact subset of  $[-r_0, +r_0]$  which contains all the possible critical points of the phases. Observe now that because  $F$  is injective and continuous there exists  $\eta > 0$  such that

$$\mathcal{B} = \tilde{F}(K \setminus (D(-1, \eta) \cup D(+1, \eta))),$$

where  $D(z, \eta) := \{w \mid |w| = 1 : |z - w| < \eta\}$ . Since  $F$  is smooth away from  $-1$  and  $+1$ , we deduce that

$$\dim_H(\mathcal{B}) \leq \dim_H(K),$$

where  $\dim_H$  stands for the Hausdorff dimension. Because we have

$$K = g^{-1}\left(\overline{\bigcup_{\gamma \in \Gamma} \gamma(\xi)}\right)$$

and since the set of accumulation points of the orbit  $\Gamma \cdot \xi$  is exactly the limit set  $\Lambda(\Gamma)$ , we deduce that

$$\dim_H(\mathcal{B}) \leq \delta(\Gamma) < 1.$$

As a consequence,  $[-r_0, +r_0] \setminus \mathcal{B}$  has non empty interior. We therefore pick  $J \subset [-r_0, +r_0]$  an interval such that

$$\bar{J} \cap \mathcal{B} = \emptyset.$$

On this interval  $J$  all points are uniformly away from the "bad critical set"  $\mathcal{B}$ . This will allow us to use the following version of non-stationary phase estimate.

**Lemma 3.2.** *Let  $I$  be a compact interval. Let  $\Phi \in C^2(I)$  and  $\varphi \in C^1(I)$ . Assume that for all  $x \in I$ ,  $\Phi'(x) \neq 0$ . Then one can find a constant  $M(I)$  such that for all  $a < b \in I$ , for all  $\lambda \geq 1$  we have*

$$\left| \int_a^b e^{i\lambda\Phi(x)} \varphi(x) dx \right| \leq M \frac{\max(1, \|\Phi''\|_{C^0(I)}) \|\varphi\|_{C^1(I)}}{\lambda (\inf_I |\Phi'|)^2}.$$

The proof of this fact is elementary: just integrate by parts. We now apply the above non-stationary principle to each term in the sum

$$\sum_{\gamma \in \Gamma} \int_{\alpha}^{\beta} e^{(\frac{1}{2} + i\lambda) B_{\xi}(0, \gamma g(r))} dr.$$

We use the fact that for all  $r \in J$ ,

$$\left| \frac{d}{dr} (B_{\xi}(0, \gamma g(r))) \right| \geq C(J) > 0,$$

and apply Lemma 2.1 to deduce that uniformly in  $\gamma$ ,

$$\int_{\alpha}^{\beta} e^{(\frac{1}{2} + i\lambda) B_{\xi}(0, \gamma g(r))} dr = O\left(\frac{e^{\frac{1}{2}d(0, \gamma 0)}}{\lambda}\right).$$

One has also to check that  $\|\frac{d^2}{dr^2} (B_{\xi}(0, \gamma g(r)))\|_{C^0(J)}$  is uniformly bounded from above, which follows easily from the formula (4). The end of the proof follows from convergence of Poincaré Series.  $\square$

**3.2. The condition of  $\xi$ -non symmetry.** We will first state the definition for the universal cover. Given two different points  $\eta_1 \neq \eta_2 \in S^1 := \partial\mathbb{H}^2$ , we denote by  $\mathcal{C}_{\eta_1, \eta_2}$  the unique (non oriented) geodesic in  $\mathbb{H}^2$  whose endpoints are  $\eta_1, \eta_2$ .

**Definition 3.3.** Let  $\xi \in \partial\mathbb{H}^2 \setminus \Lambda(\Gamma)$ . Let  $\mathcal{C}_g := \mathcal{C}_{g(-1), g(+1)} = g([-1, +1])$  be a parametrized geodesic as above. We say that  $\mathcal{C}_g$  is  $\xi$ -non symmetric ( $\xi$ -NS) iff

- (1)  $\forall \gamma_1 \neq \gamma_2 \in \Gamma$ ,  $\mathcal{C}_g$  and  $\mathcal{C}_{\gamma_1\xi, \gamma_2\xi}$  are non orthogonal.
- (2)  $\forall \gamma_1 \neq \gamma_2 \in \Gamma$ ,  $\mathcal{C}_g \neq \mathcal{C}_{\gamma_1\xi, \gamma_2\xi}$ .

**Remark.** Condition (1) implies that for all  $\gamma_1 \neq \gamma_2$ , we have

$$\operatorname{Re}(g^{-1}\gamma_1(\xi) - g^{-1}\gamma_2(\xi)) \neq 0.$$

Indeed, we cannot have  $g^{-1}\gamma_1(\xi) = g^{-1}\gamma_2(\xi)$  otherwise we would have

$$\gamma_1^{-1} \circ \gamma_2(\xi) = \xi,$$

which is impossible outside the limit set (recall that  $\xi \notin \Lambda(\Gamma)$ ). Therefore

$$\operatorname{Re}(g^{-1}\gamma_1(\xi) - g^{-1}\gamma_2(\xi)) = 0 \Rightarrow \overline{g^{-1}(\gamma_1\xi)} = g^{-1}(\gamma_2\xi),$$

which by conformal invariance of angles implies that

$$\mathcal{C}_g \perp \mathcal{C}_{\gamma_1\xi, \gamma_2\xi}.$$

Note that condition (2) is always fulfilled if  $\mathcal{C}_g$  is a *trapped geodesic*, i.e. both endpoints  $g(1)$  and  $g(-1)$  belong to the limit set  $\Lambda(\Gamma)$ . We prove below that these conditions have full measure with respect to  $\xi$ .

**Proposition 3.4.** Let  $\mathcal{C}$  be a geodesic. Then for Lebesgue almost all  $\xi \in S^1 \setminus \Lambda(\Gamma)$ ,  $\mathcal{C}$  satisfies  $\xi$ -NS.

*Proof.* We assume that  $\mathcal{C}$  is parametrized by

$$\mathcal{C} = g([-1, +1]),$$

for some Moebius map  $g$ . First we remark that either  $g(1)$  belongs to  $\Lambda(\Gamma)$  and (2) is automatically satisfied or  $g(1) \notin \Lambda(\Gamma)$  and its orbit under the action of  $\Gamma$  is discrete in  $S^1 \setminus \Lambda(\Gamma)$ . Therefore, (2) is satisfied if we chose  $\xi$  to belong to

$$S^1 \setminus \left( \Lambda(\Gamma) \cup \bigcup_{\gamma \in \Gamma} \{\gamma g(1)\} \right),$$

which is a set of full measure in  $S^1 \setminus \Lambda(\Gamma)$ .

If (1) is violated for  $\gamma_1, \gamma_2 \in \Gamma$ , we must have

$$\overline{g^{-1}(\gamma_1\xi)} = g^{-1}(\gamma_2\xi).$$

This identity can hold for only finitely many  $\xi \in S^1$ . Indeed if  $h_1, h_2$  are two (orientation preserving) isometries of the hyperbolic disc, the equation

$$(5) \quad \overline{h_1(\xi)} = h_2(\xi)$$

has at most two solutions in  $S^1 = \partial\mathbb{H}^2$ : for orientation reasons this equality cannot hold identically on  $S^1$ , any solution of (5) is a root of a non zero polynomial of degree at most 2. We therefore have to remove from  $S^1 \setminus \Lambda(\Gamma)$  a countable set of possible solutions to make sure that (1) is satisfied. In a nutshell, both (1) and (2) are satisfied for all  $\xi \in S^1 \setminus \Lambda(\Gamma)$  except for a countable set, the proof is done.  $\square$

On the quotient surface  $X = \Gamma \backslash \mathbb{H}^2$ , the condition  $\xi$ -NS translates as follows. Given a geodesic  $\mathcal{C}$  on the surface, it satisfies  $\xi$ -NS if  $\mathcal{C}$  is *never equal or orthogonal to geodesics that start and end at  $\xi$  (at infinity)*. Indeed, geodesics that start and end at  $\xi$  are lifted on  $\mathbb{H}^2$  to geodesics whose endpoints are equal to  $\xi$ , mod  $\Gamma$ , that is geodesics of type  $\mathcal{C}_{\gamma_1\xi, \gamma_2\xi}$ , for some  $\gamma_1 \neq \gamma_2 \in \Gamma$ .

**3.3. Proof of the equidistribution result on geodesics.** The goal of this subsection is to prove the following fact, which implies straightforwardly Theorem 1.2.

**Theorem 3.5.** *Assume that  $\Gamma$  is a convex co-compact group with  $\delta(\Gamma) < \frac{1}{2}$ . Let  $\mathcal{C} = g([-1, +1])$  be a geodesic satisfying  $\xi$ -NS. Then for all  $0 < r_0 < 1$ , for all  $\varphi \in C^1([-r_0, +r_0])$ ,*

$$\lim_{\lambda \rightarrow +\infty} \int_{-r_0}^{+r_0} (F_\lambda(g(r), \xi))^2 \varphi(r) dr = \frac{1}{2} \int_{-r_0}^{+r_0} E_1(g(r), \xi) \varphi(r) dr.$$

*Proof.* We start by writing

$$(F_\lambda(z, \xi))^2 = \frac{1}{2} |E_{1/2+i\lambda}(z, \xi)|^2 + \frac{1}{2} \operatorname{Re} \left( (E_{1/2+i\lambda}(z, \xi))^2 \right),$$

so that we have to investigate

$$\begin{aligned} \int_{-r_0}^{+r_0} (F_\lambda(g(r), \xi))^2 \varphi(r) dr &= \frac{1}{2} \underbrace{\int_{-r_0}^{+r_0} |E_{1/2+i\lambda}(g(r), \xi)|^2 \varphi(r) dr}_{I_1(\lambda)} \\ &\quad + \frac{1}{2} \operatorname{Re} \left( \underbrace{\int_{-r_0}^{+r_0} (E_{1/2+i\lambda}(g(r), \xi))^2 \varphi(r) dr}_{I_2(\lambda)} \right). \end{aligned}$$

We will first analyze  $I_1(\lambda)$ . By uniform convergence we can write

$$I_1(\lambda) = \sum_{\gamma_1, \gamma_2 \in \Gamma} \int_{-r_0}^{+r_0} e^{\frac{1}{2}(B_\xi(0, \gamma_1 g(r)) + B_\xi(0, \gamma_2 g(r)))} e^{i\lambda \Phi_{\gamma_1, \gamma_2}(r)} \varphi(r) dr,$$

where

$$\Phi_{\gamma_1, \gamma_2}(r) = B_\xi(0, \gamma_1 g(r)) - B_\xi(0, \gamma_2 g(r)).$$

Writing

$$\Phi_{\gamma_1, \gamma_2}(r) = B_\xi(0, \gamma_1 g(0)) - B_\xi(0, \gamma_2 g(0)) + B_{g^{-1}\gamma_1^{-1}\xi}(0, r) - B_{g^{-1}\gamma_2^{-1}\xi}(0, r),$$

we deduce that

$$\frac{d}{dr} (\Phi_{\gamma_1, \gamma_2}(r)) = 2 \frac{\operatorname{Re}(g^{-1}\gamma_2^{-1}\xi - g^{-1}\gamma_1^{-1}\xi)(r^2 - 1)}{|r - g^{-1}\gamma_2^{-1}\xi|^2 |r - g^{-1}\gamma_1^{-1}\xi|^2},$$

and therefore,

$$\inf_{[-r_0, +r_0]} \left| \frac{d}{dr} (\Phi_{\gamma_1, \gamma_2}(r)) \right| \geq \frac{1 - r_0^2}{16} |\operatorname{Re}(g^{-1}\gamma_2^{-1}\xi - g^{-1}\gamma_1^{-1}\xi)|.$$

Because we are assuming property  $\xi$ -NS, part (1), we know that for all  $\gamma_1 \neq \gamma_2$  this lower bound cannot vanish. This will allow us to apply the non-stationary phase Lemma 3.2 to the off-diagonal sums above. More precisely, we have

$$\begin{aligned} I_1(\lambda) &= \sum_{\gamma \in \Gamma} \int_{-r_0}^{+r_0} e^{(B_\xi(0, \gamma g(r)))} \varphi(r) dr \\ &\quad + \sum_{\gamma_1 \neq \gamma_2} \int_{-r_0}^{+r_0} e^{\frac{1}{2}(B_\xi(0, \gamma_1 g(r)) + B_\xi(0, \gamma_2 g(r)))} e^{i\lambda \Phi_{\gamma_1, \gamma_2}(r)} \varphi(r) dr, \end{aligned}$$

where we can write again by uniform convergence

$$\sum_{\gamma \in \Gamma} \int_{-r_0}^{+r_0} e^{(B_\xi(0, \gamma g(r)))} \varphi(r) dr = \int_{-r_0}^{+r_0} E_1(g(r), \xi) \varphi(r) dr.$$

It is important to notice that  $E_1(z, \xi)$  is a *positive non vanishing Harmonic function* on the unit disc which satisfy the trivial lower bound (given by the identity term in the sum):

$$E_1(z, \xi) \geq \frac{1 - |z|^2}{|z - \xi|^2}.$$

To complete the asymptotic analysis of  $I_1(\lambda)$ , we therefore have to show that the off-diagonal contribution goes to zero as  $\lambda$  goes to infinity. Let us write

$$\sum_{\gamma_1 \neq \gamma_2} \int_{-r_0}^{+r_0} e^{\frac{1}{2}(B_\xi(0, \gamma_1 g(r)) + B_\xi(0, \gamma_2 g(r)))} e^{i\lambda \Phi_{\gamma_1, \gamma_2}(r)} \varphi(r) dr = \sum_{\gamma_1 \neq \gamma_2} I_{\gamma_1, \gamma_2}(\lambda).$$

By Lemma 2.1, we have uniformly in  $\lambda$ ,

$$(6) \quad |I_{\gamma_1, \gamma_2}(\lambda)| \leq C(r_0) e^{-\frac{1}{2}d(0, \gamma_1 0) - \frac{1}{2}d(0, \gamma_2 0)}$$

while by the above analysis of phases  $\Phi_{\gamma_1, \gamma_2}$  and Lemma 3.2, we do have for all  $\gamma_1 \neq \gamma_2$ ,

$$(7) \quad |I_{\gamma_1, \gamma_2}(\lambda)| = O\left(\frac{1}{\lambda}\right).$$

Because we have

$$\sum_{\gamma_1, \gamma_2} e^{-\frac{1}{2}d(0, \gamma_1 0) - \frac{1}{2}d(0, \gamma_2 0)} < +\infty$$

we can deduce that

$$\lim_{\lambda \rightarrow +\infty} \sum_{\gamma_1 \neq \gamma_2} I_{\gamma_1, \gamma_2}(\lambda) = 0.$$

Indeed, fix  $\epsilon > 0$ , and choose  $T$  so large that

$$C(r_0) \times \sum_{\substack{\gamma_1 \neq \gamma_2 \\ d(0, \gamma_1 0) \geq T \text{ or } d(0, \gamma_2 0) \geq T}} e^{-\frac{1}{2}d(0, \gamma_1 0) - \frac{1}{2}d(0, \gamma_2 0)} \leq \frac{\epsilon}{2},$$

where  $C(r_0)$  is the constant in estimate (6). Writing

$$\left| \sum_{\gamma_1 \neq \gamma_2} I_{\gamma_1, \gamma_2}(\lambda) \right| \leq \frac{\epsilon}{2} + \sum_{\substack{\gamma_1 \neq \gamma_2 \\ d(0, \gamma_1 0) < T \text{ and } d(0, \gamma_2 0) < T}} |I_{\gamma_1, \gamma_2}(\lambda)|,$$

using (7), we can choose  $\lambda_0$  so large that for all  $\lambda$  with  $\lambda \geq \lambda_0$ ,

$$\sum_{\substack{\gamma_1 \neq \gamma_2 \\ d(0, \gamma_1 0) < T \text{ and } d(0, \gamma_2 0) < T}} |I_{\gamma_1, \gamma_2}(\lambda)| \leq \frac{\epsilon}{2},$$

and we are done.

Next we move on to the analysis of  $I_2(\lambda)$ . Again using uniform convergence, we have

$$\int_{-r_0}^{+r_0} (E_{1/2+i\lambda}(g(r), \xi))^2 \varphi(r) dr = \sum_{\gamma_1, \gamma_2} J_{\gamma_1, \gamma_2}(\lambda),$$

where

$$J_{\gamma_1, \gamma_2}(\lambda) = \int_{-r_0}^{+r_0} e^{\frac{1}{2}(B_\xi(0, \gamma_1 g(r)) + B_\xi(0, \gamma_2 g(r)))} e^{i\lambda \Theta_{\gamma_1, \gamma_2}(r)} \varphi(r) dr,$$

with

$$\Theta_{\gamma_1, \gamma_2}(r) = B_\xi(0, \gamma_1 g(r)) + B_\xi(0, \gamma_2 g(r)).$$

Using the same tricks as above, one can compute

$$\frac{d}{dr} (\Theta_{\gamma_1, \gamma_2}(r)) = 2 \frac{r^2 a_{\gamma_1} - 2r + a_{\gamma_1}}{(1-r^2)|r - g^{-1}\gamma_1^{-1}\xi|^2} + 2 \frac{r^2 a_{\gamma_2} - 2r + a_{\gamma_2}}{(1-r^2)|r - g^{-1}\gamma_2^{-1}\xi|^2}$$

$$= \frac{2}{1-r^2} \frac{P_{\gamma_1, \gamma_2}(r)}{|r - g^{-1}\gamma_2^{-1}\xi|^2 |r - g^{-1}\gamma_1^{-1}\xi|^2},$$

where

$$P_{\gamma, \gamma'}(r) = (a_\gamma + a_{\gamma'})r^4 - 4(a_\gamma a_{\gamma'} + 1)r^3 + 2(a_\gamma + a_{\gamma'})r^2 - 4(a_\gamma a_{\gamma'} + 1)r + a_\gamma + a_{\gamma'},$$

and  $a_\gamma = \operatorname{Re}(g^{-1}\gamma^{-1}\xi)$ . Therefore we get the lower bound

$$\left| \frac{d}{dr} (\Theta_{\gamma_1, \gamma_2}(r)) \right| \geq \frac{1}{8} |P_{\gamma_1, \gamma_2}(r)|.$$

A key observation is that this *polynomial has always degree 3 or 4*. Indeed, if we have

$$a_{\gamma_1} + a_{\gamma_2} = 0 \text{ and } a_{\gamma_1} a_{\gamma_2} + 1 = 0,$$

then  $(a_{\gamma_1}, a_{\gamma_2}) \in \{(1, -1); (-1, 1)\}$ , which would mean that either

$$\gamma_1^{-1}\xi = g(-1), \quad \gamma_2^{-1}\xi = g(1)$$

or

$$\gamma_1^{-1}\xi = g(1), \quad \gamma_2^{-1}\xi = g(-1).$$

This is not possible because of condition (2) in  $\xi$ -NS. To conclude the proof, we will need the following Van der Corput's style Lemma, to deal with the possibly highly degenerated stationary phases.

**Lemma 3.6.** *Let  $I$  be a compact non trivial interval and  $F \in C^2(I)$ ,  $\varphi \in C^1(I)$ . Assume that for all  $x \in I$ , we have*

$$|F'(x)| \geq C|P(x)|,$$

where  $P(x)$  is a polynomial of degree  $d > -\infty$ . Then as  $\lambda$  goes to infinity, we have

$$\int_I e^{i\lambda F(x)} \varphi(x) dx = O\left(\lambda^{-\frac{1}{2d+1}}\right).$$

Here we prove Lemma 3.6. Let  $P(x) = a_0 + a_1x + \dots + a_dx^d$ , with  $a_d \neq 0$ . Let  $x_1, x_2, \dots, x_d \in \mathbb{C}$  be the roots of  $P(x)$  so that we can write

$$(8) \quad P(x) = a_d(x - x_1) \dots (x - x_d).$$

Let  $\epsilon > 0$  to be specified later on. For all  $\epsilon > 0$  small enough, set

$$I(\epsilon) := \{x \in I : \forall j = 1, \dots, d, |x - x_j| \geq \epsilon\}.$$

Then for all  $\epsilon > 0$  small enough  $I_\epsilon$  is a finite union of closed intervals

$$I(\epsilon) = \bigcup_{\ell=1}^{d'} I_\ell(\epsilon),$$

with  $d' \leq d$  independent of  $\epsilon$ . On each interval  $I_\ell(\epsilon)$ ,  $F'$  does not vanish so that we can integrate by parts

$$\begin{aligned} \int_{I_\ell(\epsilon)} e^{i\lambda F(x)} \varphi(x) dx &= \frac{1}{i\lambda} \left[ e^{i\lambda F(x)} \frac{\varphi(x)}{F'(x)} \right]_{\partial I_\ell(\epsilon)} \\ &\quad - \frac{1}{i\lambda} \int_{I_\ell(\epsilon)} e^{i\lambda F(x)} \frac{d}{dx} \left( \frac{\varphi(x)}{F'(x)} \right) dx. \end{aligned}$$

Notice that by (8), we have for all  $x \in I(\epsilon)$ ,

$$|F'(x)| \geq C|a_d|\epsilon^d,$$

which yields for all  $\lambda \geq 1$  and all  $\epsilon$  small,

$$\left| \int_{I(\epsilon)} e^{i\lambda F(x)} \varphi(x) dx \right| \leq \frac{\tilde{C}}{\lambda \epsilon^{2d}},$$

where  $\tilde{C}$  is independent of  $\lambda, \epsilon$ . Writing

$$\begin{aligned} \int_I e^{i\lambda F(x)} \varphi(x) dx &= \int_{I(\epsilon)} e^{i\lambda F(x)} \varphi(x) dx + \int_{I \setminus I(\epsilon)} e^{i\lambda F(x)} \varphi(x) dx \\ &= O(\epsilon) + O\left(\frac{1}{\lambda \epsilon^{2d}}\right), \end{aligned}$$

we then choose

$$\epsilon = \lambda^{-\frac{1}{2d+1}},$$

and the proof is done.  $\square$

Note that the rate of decay as estimated above is far from being optimal, but enough for our purpose since we do not target any specific convergence rate. We can now finish the proof of the equidistribution theorem. By Lemma 2.1, we have uniformly in  $\lambda$ ,

$$|J_{\gamma_1, \gamma_2}(\lambda)| \leq C(r_0) e^{-\frac{1}{2}d(0, \gamma_1 0) - \frac{1}{2}d(0, \gamma_2 0)}$$

while Lemma 3.6 and the computation of  $\Theta'_{\gamma_1, \gamma_2}(r)$  above show that individually as  $\lambda$  goes to  $+\infty$ ,

$$|J_{\gamma_1, \gamma_2}(\lambda)| = O\left(\lambda^{-\frac{1}{9}}\right).$$

The same arguments as above then yield

$$\lim_{\lambda \rightarrow +\infty} I_2(\lambda) = 0,$$

finishing the proof of Theorem 3.5.  $\square$

**3.4. Equidistribution on real analytic curves.** In this section, we explain in a nutshell how the above equidistribution theorem on geodesics can be extended to all *real analytic* curves, for almost all  $\xi$ . The ideas are very similar to the above proof, but the price to pay to obtain a result at this level of generality is that the generic conditions on  $\xi$  have no longer a simple geometric interpretation as in the  $\xi$ -NS statement. We have chosen to include details on this generalization because it could be useful in other situations, but for the applications we have in mind (§4 and §5), geodesics are enough.

Without loss of generality, we will assume that  $g$  is a Möebius map of the unit disc and that  $\ell : [-r_0, +r_0] \rightarrow \mathbb{H}^2$  is a *real analytic* complex valued map with  $\ell(0) = 0$  and  $\ell'(r) \neq 0$  for all  $r \in [-r_0, +r_0]$ . We will consider the map

$$g \circ \ell : [-r_0, +r_0] \rightarrow \mathbb{H}^2$$

as a parametrized curve on which we want to prove the same statement as above. Following the exact same lines, we need to analyze the two phase functions

$$\Phi_{\gamma_1, \gamma_2}(r) = B_\xi(0, \gamma_1 g(\ell(r))) - B_\xi(0, \gamma_2 g(\ell(r))),$$

$$\Theta_{\gamma_1, \gamma_2}(r) = B_\xi(0, \gamma_1 g(\ell(r))) + B_\xi(0, \gamma_2 g(\ell(r))).$$

Carrying the same computations as in the geodesic case, we have

$$\Phi'_{\gamma_1, \gamma_2}(r) = -2\operatorname{Re} \left( \frac{\ell'(r)(g^{-1}\gamma_2^{-1}\xi - g^{-1}\gamma_1^{-1}\xi)}{(\ell(r) - g^{-1}\gamma_2^{-1}\xi)(\ell(r) - g^{-1}\gamma_1^{-1}\xi)} \right).$$

We will show that  $\Phi'_{\gamma_1, \gamma_2}$  is non identically vanishing for generic  $\xi$ . Evaluating the above formula at  $r = 0$  yields

$$\Phi'_{\gamma_1, \gamma_2}(0) = -2\operatorname{Re} \left( \overline{\ell'(0)}(g^{-1}\gamma_2^{-1}\xi - g^{-1}\gamma_1^{-1}\xi) \right).$$

Since we are assuming  $\gamma_1 \neq \gamma_2$  we can use the exact same ideas as before to show that

$$\Phi'_{\gamma_1, \gamma_2}(0) \neq 0$$

for a set of  $\xi$  with full measure in the discontinuity set. Being a real-analytic, non identically vanishing function,  $\Phi'_{\gamma_1, \gamma_2}(r)$  has a holomorphic extension to an open complex domain

$$[-r_0, +r_0] \subset \Omega \subset \mathbb{C}$$

and by further shrinking  $\Omega$  we can assume that it has finitely many zeros  $z_1, \dots, z_d$  (repeated with multiplicity) in  $\Omega$ . The map

$$z \mapsto \frac{\Phi'_{\gamma_1, \gamma_2}(z)}{\prod_{j=1}^d (z - z_j)}$$

is holomorphic, non vanishing on  $\Omega$  and therefore there exists  $C > 0$  such that for all  $r \in [-r_0, +r_0]$ ,

$$|\Phi'_{\gamma_1, \gamma_2}(r)| \geq C \left| \prod_{j=1}^d (z - z_j) \right|.$$

We can then apply Lemma 3.6 to show that for generic  $\xi$ , all  $\gamma_1 \neq \gamma_2$

$$\lim_{\lambda \rightarrow +\infty} I_{\gamma_1, \gamma_2}(\lambda) = 0.$$

We now need to treat the second phase function  $\Theta_{\gamma_1, \gamma_2}(r)$ . Performing similar calculations we have

$$\Theta'_{\gamma_1, \gamma_2}(r) = 2\operatorname{Re} \left( \ell'(r) \frac{|\ell(r)|^2(\xi_1 + \xi_2) - 2\ell(r) - 2\overline{\ell(r)}\xi_1\xi_2 + \xi_1 + \xi_2}{(1 - |\ell(r)|^2)(\ell(r) - \xi_1)(\ell(r) - \xi_2)} \right),$$

where we have set for simplicity  $\xi_1 = g^{-1}\gamma_1^{-1}\xi$ ,  $\xi_2 = g^{-1}\gamma_2^{-1}\xi$ . We obtain for  $r = 0$ ,

$$\Theta'_{\gamma_1, \gamma_2}(0) = 2\operatorname{Re} \left( \overline{\ell'(0)}(g^{-1}\gamma_2^{-1}\xi + g^{-1}\gamma_1^{-1}\xi) \right).$$

We want to show once again that for a generic choice of  $\xi$ , this is not 0. First, remark that we cannot have for all  $\xi \in S^1$

$$g^{-1}\gamma_2^{-1}\xi = -g^{-1}\gamma_1^{-1}\xi.$$

Indeed, such an identity would imply (by analytic continuation) that for all  $z \in \mathbb{H}^2$ ,

$$g^{-1}\gamma_2^{-1}\gamma_1 g(z) = -z.$$

If  $\gamma_1 = \gamma_2$  we clearly have a contradiction while if  $\gamma_1 \neq \gamma_2$  this formula would show that  $\gamma_2^{-1}\gamma_1$  is an *elliptic isometry*, simply because it is conjugated to  $z \mapsto -z$ , which is elliptic. Because  $\Gamma$  is a convex co-compact group whose elements are all hyperbolic (except identity), we have again a contradiction. Therefore

$$g^{-1}\gamma_2^{-1}\xi = -g^{-1}\gamma_1^{-1}\xi$$

can hold for at most two points in  $S^1$ . By removing a countable set of the discontinuity domain, we can rule out this case. We are left with the case

$$\overline{\ell'(0)}g^{-1}\gamma_2^{-1}\xi = -\ell'(0)\overline{g^{-1}\gamma_1^{-1}\xi},$$

which can be treated as in the previous section by using an orientation argument. Discarding another countable set of points, we can make sure that for all  $\gamma_1, \gamma_2 \in \Gamma$ ,

$$\Theta'_{\gamma_1, \gamma_2}(0) \neq 0.$$

We can then use the same arguments as before and apply Lemma 3.6 to get decay of oscillatory integrals

$$\lim_{\lambda \rightarrow +\infty} J_{\gamma_1, \gamma_2}(\lambda) = 0.$$

To conclude this section we point out that it is unclear to us whether Proposition 3.1 holds for general analytic curves, which prevents us from extending the lower bound of Theorem 1.1 to analytic curves. However as pointed out in the next

section, it works without major modification for the upper bound, extending the upper bound of Theorem 1.1 to real-analytic curves.

#### 4. COUNTING INTERSECTIONS OF NODAL LINES WITH GEODESICS

In this section we prove Theorem 1.1, using the previous equidistribution result. We assume that  $\mathcal{C} = g([-1, +1])$  is a fixed geodesic satisfying  $\xi$ -NS, and that  $\delta(\Gamma) < \frac{1}{2}$  as we did before.

**4.1. The lower bound.** Let  $\mathcal{C}_0 \subset \mathcal{C}$  be a geodesic segment given by  $\mathcal{C}_0 = g([-r_0, +r_0])$ . We pick  $J \subset [-r_0, +r_0]$  so that the conclusion of Proposition 3.1 holds. Since we have

$$\int_J |F_\lambda(g(r), \xi)| dr \geq \|F_\lambda\|_{L^\infty(g(J))}^{-1} \int_J (F_\lambda(g(r), \xi))^2 dr,$$

remembering the bound (3) we can use Theorem 3.5 which says that

$$\lim_{\lambda \rightarrow \infty} \int_J (F_\lambda(g(r), \xi))^2 dr = \frac{1}{2} \int_J E_1(g(r), \xi) dr,$$

to conclude that one can find  $C > 0$  such that for all  $\lambda$  large,

$$\int_J |F_\lambda(g(r), \xi)| dr \geq C.$$

Let  $N(\lambda) \geq 0$  be the number of zeros of  $r \mapsto F_\lambda(g(r), \xi)$  in the interval  $\text{Int}(J)$ . By writing

$$J = \bigcup_{\ell=0}^{N(\lambda)} J_\ell,$$

where  $r \mapsto F_\lambda(g(r), \xi)$  has constant sign on each  $J_\ell$ , we deduce by Proposition 3.1 that

$$\begin{aligned} 0 < C &\leq \int_J |F_\lambda(g(r), \xi)| dr = \sum_{\ell=0}^{N(\lambda)} \left| \int_{J_\ell} F_\lambda(g(r), \xi) dr \right| \\ &\leq (N(\lambda) + 1) \sup_{\alpha < \beta \in J} \left| \int_\alpha^\beta F_\lambda(g(r), \xi) dr \right| \leq \frac{\tilde{C}(N(\lambda) + 1)}{\lambda}, \end{aligned}$$

which implies that for all  $\lambda$  large enough,

$$N(\lambda) \geq C' \lambda,$$

and the proof of the lower bound is done.

**4.2. The upper bound.** As we said in the introduction, we will need to use analyticity to prove the upper bound on the number of intersection of nodal lines with geodesics. We will therefore start by proving the following fact, which is a way to "complexify" restrictions of eigenfunctions  $F_\lambda$  to geodesics.

**Proposition 4.1.** *Let  $\mathcal{C} = g([-1, +1])$  be a geodesic. Then for all  $\lambda \in \mathbb{R}$ , the map  $z \mapsto F_\lambda(g(z), \xi)$ , defined on  $(-1, +1)$ , admits a holomorphic extension to the unit disc  $\mathbb{D}$ , which is denoted by  $\tilde{F}_{\lambda, g}(z, \xi)$ . Moreover, for all compact subset  $K \subset \mathbb{D}$ , there exist  $\beta_K, C_K > 0$  such that for all  $\lambda \geq 0$ , we have*

$$\sup_{z \in K} |\tilde{F}_{\lambda, g}(z, \xi)| \leq C_K e^{\beta_K \lambda}.$$

*Proof.* We recall that for all  $r \in (-1, +1)$ , we have the convergent series expansion

$$F_\lambda(g(r), \xi) = \sum_{\gamma \in \Gamma} e^{\frac{1}{2}B_\xi(0, \gamma g(r))} \cos(\lambda B_\xi(0, \gamma g(r))).$$

Since we have

$$B_\xi(0, \gamma g(r)) = B_\xi(0, \gamma g(0)) + B_{g^{-1}\gamma^{-1}\xi}(0, r),$$

it is enough to continue analytically

$$r \mapsto B_{g^{-1}\gamma^{-1}\xi}(0, r) = \log \left( \frac{1 - r^2}{|r - g^{-1}\gamma^{-1}\xi|^2} \right).$$

We set for simplicity  $\eta := g^{-1}\gamma^{-1}\xi$  and for all  $z \in (-1, +1)$ ,

$$G_\eta(z) := \frac{1 - z^2}{|z - \eta|^2} = \frac{1 - z^2}{(z - \eta)(z - \bar{\eta})}.$$

Clearly  $G_\eta(z)$  extends holomorphically to the unit disc  $\mathbb{D}$ , where it does not vanish. We can therefore define a complex logarithm by setting for all  $z \in \mathbb{D}$

$$(9) \quad \mathbb{L}(G_\eta)(z) := \int_0^z \frac{G'_\eta(\zeta)}{G_\eta(\zeta)} d\zeta = z \int_0^1 \frac{G'_\eta(zt)}{G_\eta(zt)} dt.$$

We obtain a holomorphic function  $\mathbb{L}(G_\eta)(z)$  on  $\mathbb{D}$  which has the following properties:

- $\forall r \in (-1, +1)$ ,  $\mathbb{L}(G_\eta)(r) = \log G_\eta(r) = B_\eta(0, r)$ .
- $\forall z \in \mathbb{D}$ ,  $e^{\mathbb{L}(G_\eta)(z)} = G_\eta(z)$ .

By using formula (9), one can check that for all  $0 < r_1 < 1$ ,

$$\sup_{|z| \leq r_1} |\mathbb{L}(G_\eta)(z)| \leq C(r_1),$$

where  $C(r_1)$  is uniform in  $\eta := g^{-1}\gamma^{-1}\xi$ . Writing

$$\begin{aligned} & \cos(\lambda B_\xi(0, \gamma g(0)) + \lambda \mathbb{L}(G_\eta)(z)) \\ &= \cos(\lambda B_\xi(0, \gamma g(0))) \cos(\lambda \mathbb{L}(G_\eta)(z)) - \sin(\lambda B_\xi(0, \gamma g(0))) \sin(\lambda \mathbb{L}(G_\eta)(z)), \end{aligned}$$

and using the bounds for all  $z \in \mathbb{C}$ ,

$$|\cos(z)| \leq 2e^{|\operatorname{Im}(z)|}, \quad |\sin(z)| \leq 2e^{|\operatorname{Im}(z)|},$$

we deduce that for all  $|z| \leq r_1$  and  $\lambda \geq 0$ ,

$$|\cos(\lambda B_\xi(0, \gamma g(0)) + \lambda \mathbb{L}(G_\eta)(z))| \leq \tilde{C}(r_1) e^{\beta r_1 \lambda}.$$

Combining this last bound with Lemma 2.1 shows uniform convergence on

$$\{|z| \leq r_1\}$$

of the above series, hence holomorphy and the claimed bound.  $\square$

Notice that for a more general real-analytic curve, a similar statement follows straightforwardly, with the difference that it will hold on a smaller domain  $\Omega \subset \mathbb{D}$ .

The rest of the proof of the upper bound on the number of intersections of nodal lines with  $\mathcal{C}_0 \subset \mathcal{C}$  will follow from Theorem 3.5 combined with Jensen's formula. The version of Jensen's formula we will use is the following.

**Proposition 4.2.** *Let  $f$  be a holomorphic function on the open disc  $D(w, R)$ , and assume that  $f(w) \neq 0$ . Let  $N_f(r)$  denote the number of zeros of  $f$  in the closed disc  $\bar{D}(w, r)$ . For all  $\tilde{r} < r < R$ , we have*

$$N_f(\tilde{r}) \leq \frac{1}{\log(r/\tilde{r})} \left( \frac{1}{2\pi} \int_0^{2\pi} \log |f(w + re^{i\theta})| d\theta - \log |f(w)| \right).$$

For a reference on Jensen's formula, we refer the reader to the classics, for example Titchmarsh [20]. Let  $C_0 = g([-r_0, +r_0])$  be a geodesic segment as above, with  $0 < r_0 < 1$ . Fix  $\epsilon > 0$  so small that  $r_0 + 3\epsilon < 1$  and set

$$r_1 = r_0 + \epsilon, \quad r_2 = r_0 + 2\epsilon, \quad r_3 = r_0 + 3\epsilon.$$

If  $D(w, r)$  denotes the complex open disc with center  $w$  and radius  $r$ , we then have for all  $x \in [-\epsilon, +\epsilon]$

$$D(0, r_0) \subset D(x, r_1) \subset D(x, r_2) \subset \overline{D(0, r_3)} \subset \mathbb{D}.$$

Let  $N(\lambda)$  denote the number of zeros of  $r \mapsto F_\lambda(g(r), \xi)$  in the interval  $[-r_0, +r_0]$ . By applying Theorem 3.5 on the short interval  $[-\epsilon, +\epsilon]$ , we have

$$\lim_{\lambda \rightarrow +\infty} \int_{-\epsilon}^{+\epsilon} (F_\lambda(g(r), \xi))^2 dr = \frac{1}{2} \int_{-\epsilon}^{+\epsilon} E_1(g(r), \xi) dr,$$

which shows that for all  $\lambda$  large enough we have

$$0 < C_\epsilon := \frac{1}{2} \left( \frac{1}{2\epsilon} \int_{-\epsilon}^{+\epsilon} E_1(g(r), \xi) dr \right)^{1/2} \leq \sup_{r \in [-\epsilon, +\epsilon]} |F_\lambda(g(r), \xi)|.$$

For all  $\lambda$  large, we denote by  $x_\lambda \in [-\epsilon, +\epsilon]$  a point such that

$$|F_\lambda(g(x_\lambda), \xi)| = \sup_{r \in [-\epsilon, +\epsilon]} |F_\lambda(g(r), \xi)|.$$

Applying Jensen's formula to  $\tilde{F}_{\lambda, g}(z, \xi)$  on  $D(x_\lambda, r_1) \subset D(x_\lambda, r_2)$ , we have

$$\begin{aligned} N(\lambda) &\leq \\ &\frac{1}{\log(r_2/r_1)} \left( \frac{1}{2\pi} \int_0^{2\pi} \log |\tilde{F}_{\lambda, g}(x_\lambda + r_2 e^{i\theta}, \xi)| d\theta - \log |F_\lambda(g(x_\lambda), \xi)| \right) \\ &\leq \frac{1}{\log(r_2/r_1)} \left( \sup_{|z| \leq r_3} \log |\tilde{F}_{\lambda, g}(z, \xi)| + \log(C_\epsilon^{-1}) \right). \end{aligned}$$

Using the estimate of Proposition 4.1, we then deduce that as  $\lambda$  goes to infinity,

$$N(\lambda) = O(\lambda),$$

and the proof is completed.  $\square$

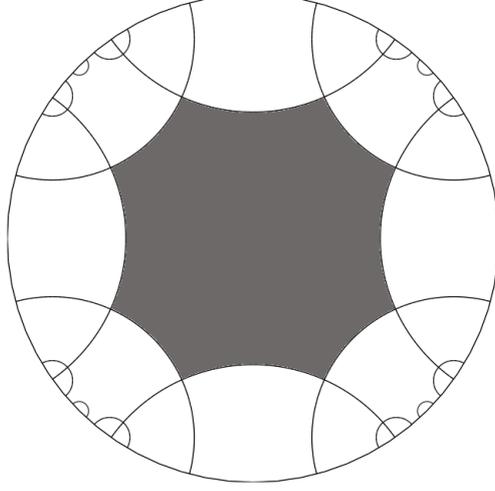
To deal with more general real-analytic curves which extend holomorphically to a smaller domain  $\Omega \subset \mathbb{D}$ , we just need to replicate the same argument with several discs instead of a single one. We omit it for simplicity.

## 5. COUNTING NODAL DOMAINS

**5.1. The upper bound in  $X_0$ .** Let us introduce some notations. We will work on the universal cover  $\mathbb{H}^2$ , so that the convex core  $X_0$  is the image under the covering  $\mathbb{H}^2 \rightarrow \Gamma \backslash \mathbb{H}^2$  of a compact geodesic polygon

$$\mathcal{P} \subset \mathbb{H}^2.$$

The polygon  $\mathcal{P}$  has finitely many sides which are geodesic segments, see the picture below for an example of such a polygon in  $\mathbb{H}^2 = \mathbb{D}$ , the gray hyperbolic octagon is  $\mathcal{P}$ .



We choose  $\xi \in S^1 \setminus \Lambda(\Gamma)$  such that the upper bound of Theorem 1.1 is valid on the full boundary  $\partial\mathcal{P}$ , which can be done for a set of full measure. We recall that the nodal domains of  $F_\lambda(z, \xi) : \mathbb{H}^2 \rightarrow \mathbb{R}$  are by definition the connected components of

$$\mathbb{H}^2 \setminus \{F_\lambda(z, \xi) = 0\}.$$

The nodal domains  $\mathcal{D}$  which do intersect  $\mathcal{P}$  fall into two categories. Either

$$\overline{\mathcal{D}} \cap \partial\mathcal{P} \neq \emptyset,$$

and thanks to Theorem 1.1 there are at most  $O(\lambda)$  of them, or

$$\overline{\mathcal{D}} \subset \text{Int}(\mathcal{P}).$$

Notice that in that case, since  $F_\lambda$  has constant sign on  $\mathcal{D}$ , the eigenvalue  $1/4 + \lambda^2$  must be the first eigenvalue of the hyperbolic Laplacian  $\Delta_{\mathbb{H}^2}$  on  $\mathcal{D}$  for the *Dirichlet boundary problem*:

$$\begin{cases} \Delta_{\mathbb{H}^2} \psi &= \mu \psi \\ \psi &= 0 \quad \text{on} \quad \partial\mathcal{D}. \end{cases}$$

Let  $\lambda_1(\mathcal{D})$  denote the smallest eigenvalue for the above Dirichlet problem. We will use the following key lower bound.

**Proposition 5.1.** *Fix  $\epsilon_0 > 0$ , then there exists  $C_0 > 0$  such that for all domain  $\Omega \subset \mathbb{H}^2$  with  $\text{Vol}(\Omega) \leq \epsilon_0$*

$$\lambda_1(\Omega) \geq \frac{C_0}{\text{Vol}(\Omega)}.$$

*Proof.* We first use the Faber-Krahn inequality for domains in  $\mathbb{H}^2$ , see Chavel [4] p. 87. It is valid on simply connected spaces of constant curvature. If  $\Omega$  is a compact domain of  $\mathbb{H}^2$  with piecewise  $C^\infty$  boundary, then the first Dirichlet eigenvalue  $\lambda_1(\Omega)$  of the Laplacian satisfies  $\lambda_1(\Omega) \geq \lambda_1(D)$ , where  $D$  is a geodesic disc with same (hyperbolic) volume. The game is now to prove a lower bound for the first eigenvalue on a disc  $D$  of the hyperbolic plane, for small values of the radius. We

use the disc model for  $\mathbb{H}^2$  and can assume that  $D = D(0, r)$  (euclidean disc) is centered at 0. By the min-max principle, we have

$$\lambda_1(D(0, r)) = \inf_{\varphi \neq 0 \in C_0^\infty(D)} \frac{\int_D \varphi(\Delta_{\mathbb{H}^2} \varphi) d\text{Vol}}{\int_D \varphi^2 d\text{Vol}}.$$

But we have

$$\int_D \varphi(\Delta_{\mathbb{H}^2} \varphi) d\text{Vol} = \int_D \varphi(\Delta \varphi) dm,$$

where  $m$  is the Lebesgue measure and  $\Delta$  the positive euclidean Laplacian, while

$$\int_D \varphi^2 d\text{Vol} = \int_D \varphi^2(z) \frac{4dm(z)}{(1-|z|^2)^2} \leq \frac{4}{(1-r_0^2)^2} \int_D \varphi^2(z) dm(z),$$

as long as  $r \leq r_0 < 1$ . We therefore have

$$\lambda_1(D(0, r)) \geq \frac{(1-r_0^2)^2}{4} \lambda_1^{\text{euc}}(D(0, r)),$$

where  $\lambda_1^{\text{euc}}$  denotes the first Dirichlet eigenvalue for the *euclidean* Laplacian. A simple change of coordinates in the min-max then shows that

$$\lambda_1^{\text{euc}}(D(0, r)) \geq \frac{\lambda_1^{\text{euc}}(D(0, 1))}{r^2}.$$

Using the formula for the hyperbolic area of  $D(0, r)$

$$\text{Vol}(\Omega) = \text{Vol}(D(0, r)) = \frac{4\pi r^2}{1-r^2}$$

shows that

$$\lambda_1(\Omega) \geq \frac{\pi(1-r_0^2)^2 \lambda_1^{\text{euc}}(D(0, 1))}{\text{Vol}(\Omega)},$$

and the claim is proved.  $\square$

Going back to the proof of the upper bound, let  $(\mathcal{D}_i)_{i \in I}$  be the (finite) collection of nodal domains  $\mathcal{D}_i$  that are inside  $\text{Int}(\mathcal{P})$ . By volume comparison, we have

$$\text{Vol}(\cup_{i \in I} \mathcal{D}_i) = \sum_{i \in I} \text{Vol}(\mathcal{D}_i) \leq \text{Vol}(\mathcal{P}).$$

Let  $J \subset I$  be the set of indexes such that for all  $j \in J$ ,  $\text{Vol}(\mathcal{D}_j) \leq \epsilon_0$ . By Proposition 5.1 we get

$$\frac{\#(J)C_0}{1/4 + \lambda^2} \leq \text{Vol}(\mathcal{P}),$$

which obviously shows that  $\#(J) = O(\lambda^2)$ . Similarly we have

$$\#(I \setminus J) \leq \epsilon_0^{-1} \text{Vol}(\mathcal{P}) = O(1).$$

As a conclusion we have shown that the total number of nodal domains that intersect  $\mathcal{P}$  is  $O(\lambda^2)$ , thus completing the proof of the upper bound.

#### APPENDIX 1: HOROCYCLIC STRUCTURE NEAR $\xi$ .

In this short appendix, we give an elementary proof of the fact that the eigenfunctions  $F_\lambda(z, \xi)$  have infinitely many *non-compact* nodal lines when  $\lambda \neq 0$ . Recall from section 2 that for all  $z \in \mathbb{D}$ , we have the convergent series ( $\delta(\Gamma) < \frac{1}{2}$ )

$$F_\lambda(z, \xi) = \sum_{\gamma \in \Gamma} e^{\frac{1}{2}B_\xi(0, \gamma z)} \cos(\lambda B_\xi(0, \gamma z)).$$

Using the cocycle property and isolating the identity term, we have

$$F_\lambda(z, \xi) = \frac{\sqrt{1-|z|^2}}{|z-\xi|} \cos(\lambda B_\xi(0, z)) + \sum_{\gamma \neq Id} e^{\frac{1}{2}B_\xi(0, \gamma 0)} \frac{\sqrt{1-|z|^2}}{|z-\gamma^{-1}\xi|} \cos(\lambda B_\xi(0, \gamma z)).$$

Fix  $\epsilon > 0$  and set

$$\mathcal{U}_\epsilon := \{z \in \mathbb{D} : 1 - |z|^2 \geq \epsilon^{-2}|z - \xi|^2\} \subset D(\xi, \epsilon),$$

which is a horoball based at  $\xi$  of euclidian size  $\epsilon$ . Observe now that for all  $z \in \mathcal{U}_\epsilon$ , for all  $\gamma \neq Id$ , the quantity  $|z - \gamma^{-1}\xi|$  is uniformly bounded away from 0, provided that we take  $\epsilon > 0$  small enough. This is again due to the fact that the orbit of  $\xi$  under  $\Gamma$ -action accumulates only on the limit set. Using convergence of Poincaré series, we deduce that one can find  $C > 0$  such that for all  $\epsilon > 0$  and  $z \in \mathcal{U}_\epsilon$ ,

$$(10) \quad \left| F_\lambda(z, \xi) - \frac{\sqrt{1 - |z|^2}}{|z - \xi|} \cos(\lambda B_\xi(0, z)) \right| \leq C\sqrt{\epsilon}.$$

Consider now

$$\mathcal{D}_\epsilon := \{z \in \mathcal{U}_\epsilon : |\cos(\lambda B_\xi(0, z))| \geq \frac{1}{2}\}.$$

by taking  $\epsilon > 0$  small enough so that

$$\frac{1}{2}\epsilon^{-1} > C\sqrt{\epsilon},$$

we can make sure that  $F_\lambda(z, \xi)$  and  $\cos(\lambda B_\xi(0, z))$  have the same sign on  $\mathcal{D}_\epsilon$ . Because we assume that  $\lambda \neq 0$  and the level sets of  $z \mapsto B_\xi(0, z)$  are horocycles based at  $\xi$  we can decompose

$$\mathcal{D}_\epsilon = \bigcup_{j=0}^{\infty} V_j$$

such that :

- Each  $V_j$  is a non-compact connected component whose boundary is the union of two horocycles.
- The sign of  $F_\lambda(z, \xi)$  is constant on each  $V_j$  and alternating with  $j$ .
- For each  $j$  we have  $V_j = \{a_j \leq \frac{1 - |z|^2}{|z - \xi|^2} \leq b_j\}$  for some "nested" sequences  $(a_j)$ ,  $(b_j)$  satisfying  $\epsilon^{-2} \leq a_j < b_j < a_{j+1} < b_{j+1} < \dots$

Now consider the nodal set

$$\mathcal{N}_\lambda(\xi) \cap \left\{ b_j < \frac{1 - |z|^2}{|z - \xi|^2} < a_{j+1} \right\}.$$

It has to be non-empty because  $F_\lambda(z, \xi)$  has opposite signs on  $V_j$  and  $V_{j+1}$  and moreover has at least one non-compact connected component (both  $V_j$  and  $V_{j+1}$  are non-compact) thus yielding the existence of infinitely many non-compact nodal lines. A similar argument also shows that there are infinitely many nodal domains in the vicinity of  $\xi$ . At the numerical level, these non compact nodal lines appear as slightly distorted horocycles as the pictures in the next appendix will show. Notice that to prove more quantitative estimates, a naive approach based on Thom's isotopy theorem would not work: identity (10) does not hold in  $C^1$  sense, simply because

$$|\nabla_z B_\xi(0, z)| = \frac{2}{1 - |z|^2}.$$

## APPENDIX 2 : NODAL PORTRAITS OF EISENSTEIN SERIES

In this appendix, we provide some numerical computations of nodal lines of Eisenstein series for surfaces  $\Gamma_\theta \backslash \mathbb{H}^2$  which are pair of pants. Let  $\theta \in (0, \pi/2)$ , and set

$$r_\theta := \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right); \quad R_\theta := \frac{1}{\cos(\pi/4 - \theta/2)}.$$

We next consider the Möebius transforms  $\gamma_1$  and  $\gamma_2$  given by

$$\gamma_1(z) := \frac{zR_\theta e^{-i\pi/4} - 1}{z - R_\theta e^{i\pi/4}}; \quad \gamma_2(z) := \frac{zR_\theta e^{-3i\pi/4} - 1}{z - R_\theta e^{3i\pi/4}}.$$

The group  $\Gamma_\theta$  generated by  $\gamma_1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1}$  is then a *convex co-compact Schottky group*. Indeed, if  $D(z, r)$  denotes the closed disc with radius  $r$  centred at  $z$ , we have ( $\widehat{\mathbb{C}}$  stands for the Riemann sphere)

$$\gamma_1(D(R_\theta e^{i\pi/4}, r_\theta)) = \widehat{\mathbb{C}} \setminus \text{Int}(D(R_\theta e^{-i\pi/4}, r_\theta)),$$

while

$$\gamma_2(D(R_\theta e^{3i\pi/4}, r_\theta)) = \widehat{\mathbb{C}} \setminus \text{Int}(D(R_\theta e^{-3i\pi/4}, r_\theta)).$$

A *fundamental domain* for the action of  $\Gamma_\theta$  on  $\mathbb{D}$  is given by the region outside the discs  $D(e^{i\pi/4+j\pi/2}, r_\theta)$ ,  $j = 0, \dots, 3$ . The associated surface  $X_\theta := \Gamma_\theta \backslash \mathbb{H}^2$  is then a pair of pants.

To compute numerically nodals lines and nodal domains, we rest on the following simple fact. Given a (reduced) word

$$\gamma = \gamma_{i_1} \circ \gamma_{i_2} \circ \dots \circ \gamma_{i_n} \in \Gamma,$$

we denote by  $n = |\gamma|$  its length.

**Proposition 5.2.** *Assume  $\delta(\Gamma) < 1/2$ , then there exists  $0 < \Upsilon < 1$  such that for all compact subset  $K \subset \mathbb{D}$  we have for all  $\lambda$ , for all  $n \geq 1$ ,*

$$\sup_{z \in K} \left| F_\lambda(z, \xi) - \sum_{|\gamma| \leq n} e^{\frac{1}{2} B_\xi(0, \gamma z)} \cos(\lambda B_\xi(0, \gamma z)) \right| \leq C_{K, \xi} \Upsilon^n.$$

*Proof.* This is an easy consequence of Lemma 2.1. First we recall that the orbital counting function  $N(T)$  defined by

$$N(T) := \#\{\gamma \in \Gamma : d(0, \gamma 0) \leq T\}$$

enjoys the general upper bound as  $T \rightarrow +\infty$

$$N(T) = O(e^{\delta T}).$$

As a consequence, a simple Stieltjes-integration by parts shows that

$$\sum_{d(0, \gamma 0) > T} e^{-\frac{1}{2} d(0, \gamma 0)} = O\left(e^{(\delta - \frac{1}{2})T}\right).$$

Applying Lemma 2.1, we have for all  $z \in K$ ,

$$\left| F_\lambda(z, \xi) - \sum_{|\gamma| \leq n} e^{\frac{1}{2} B_\xi(0, \gamma z)} \cos(\lambda B_\xi(0, \gamma z)) \right| \leq C_{k, \xi} \sum_{|\gamma| > n} e^{-\frac{1}{2} d(0, \gamma 0)}.$$

By a classical result of J. Milnor [18], there exists a constant  $M(\Gamma) > 0$  such that for all  $\gamma \in \Gamma$ , we have

$$M(\Gamma)^{-1} \leq \frac{d(0, \gamma 0)}{|n|} \leq M(\Gamma).$$

The proof is done since we have now

$$\sum_{|\gamma| > n} e^{-\frac{1}{2} d(0, \gamma 0)} \leq \sum_{d(0, \gamma 0) > n/M(\Gamma)} e^{-\frac{1}{2} d(0, \gamma 0)} \leq \tilde{C} e^{n \frac{(\delta - 1/2)}{M(\Gamma)}}. \quad \square$$

Empirically, for the  $\Gamma_\theta$ -family of groups, exponential speed of convergence allows us to restrict summation to words of length  $n \leq 3$  which leaves us with a 17 terms sum to approximate  $F_\lambda(z, \xi)$  on compact regions. For our computation to make sense, we need to work in a range of values of  $\theta$  for which we know that  $\delta(\Gamma_\theta) < \frac{1}{2}$ . By the arguments of McMullen [17], one can see that

$$\lim_{\theta \rightarrow 0} \delta(\Gamma_\theta) = 1, \quad \lim_{\theta \rightarrow \pi/2} \delta(\Gamma_\theta) = 0.$$

Using the algorithm of Jenkinson-Pollicott [9], it is possible to compute some highly accurate approximations of  $\delta$ . We will however simply rely on an effective estimate.

**Proposition 5.3.** *Using the above notations, we have for all  $\theta \geq \pi/4$ ,*

$$\delta(\Gamma_\theta) \leq \frac{\log(3)}{-2\log(\tan(\pi/4 - \theta/2)) + \log(2 - \sqrt{2})}.$$

*As a consequence, for all  $\theta \geq \frac{21\pi}{60}$ , we have  $\delta(\Gamma_\theta) < \frac{1}{2}$ .*

*Proof.* This is a standard application of Bowen's formula [3] (see also in [17] where a similar bound is derived for a different family of Schottky groups). For  $j = 1, \dots, 4$ , set

$$\mathcal{D}_j := D\left(R_\theta e^{i\pi/4 + (j-1)\pi/2}, r_\theta\right), \quad I_j := \mathcal{D}_j \cap \partial\mathbb{D}.$$

We also set  $\gamma_3 = \gamma_2^{-1}$  and  $\gamma_4 = \gamma_1^{-1}$ . The Bowen-Series map

$$T : I := \bigcup_{j=1}^4 I_j \rightarrow \partial\mathbb{D}$$

is defined by

$$T(z) := \gamma_j(z) \text{ if } z \in I_j.$$

This map is eventually uniformly expanding and the *non-wandering set* is precisely

$$\bigcap_{n \geq 0} T^{-n}(I) = \Lambda(\Gamma_\theta),$$

where  $\Lambda$  denotes the *limit set* of  $\Gamma_\theta$ . Bowen's formula states that we have

$$\delta(\Gamma_\theta) = \sup_{\mu} \frac{h_{\mu}(T)}{\int_{\Lambda} \log |T'| d\mu},$$

where the supremum is taken over all  $T$ -invariant probability measures and  $h_{\mu}$  is the metric entropy. Since we have the identity for the topological entropy

$$\sup_{\mu} h_{\mu}(T) = \log(3)$$

we can simply bound

$$\delta \leq \frac{\log(3)}{\inf_{\Lambda} \log |T'|}.$$

Using the symmetry of the fundamental domain, we are left with estimating

$$|\gamma_1'(\gamma_4(e^{i\theta/2}))| = \frac{|e^{i\theta/2} - R_\theta e^{-i\pi/4}|^2}{r_\theta^2}.$$

Because we have the crude lower bound

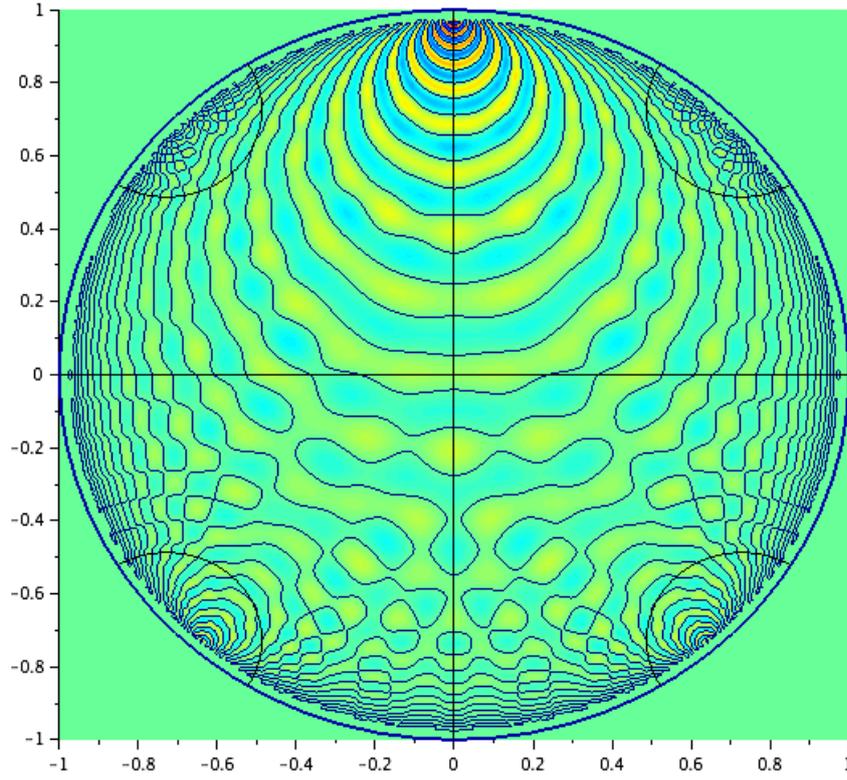
$$|e^{i\theta/2} - R_\theta e^{-i\pi/4}|^2 \geq 2 - \sqrt{2},$$

the proof is done and the bound is valid for all  $\theta \in (0, \pi/2)$  such that

$$\log(2 - \sqrt{2}) - 2\log(r_\theta) > 0,$$

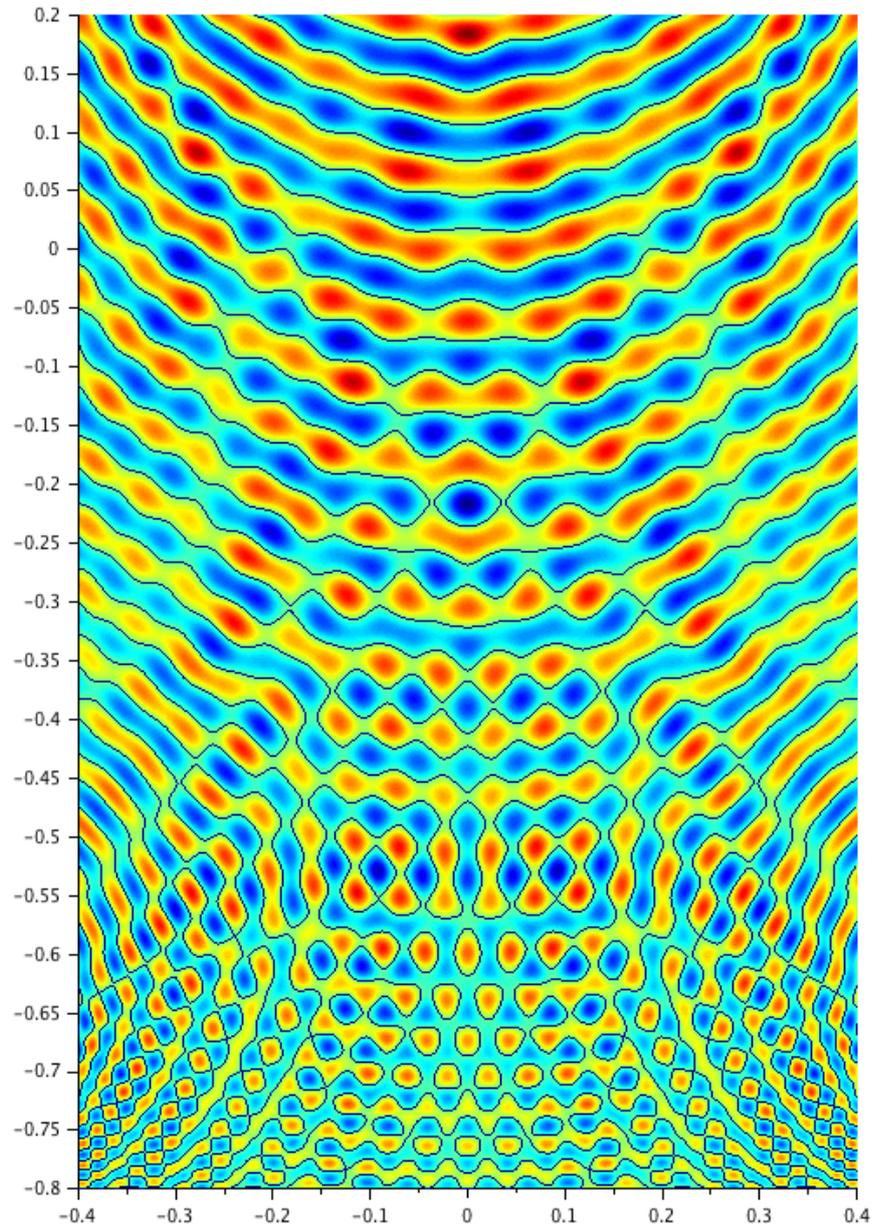
and this is the case for  $\theta \geq \pi/4$ . One can then check numerically that for  $\theta \geq \frac{21\pi}{60}$ , we have  $\delta(\Gamma_\theta) \leq 0.4668$ .  $\square$

In the next pages, we provide contour plots of the level sets of eigenfunctions  $F_\lambda(z, \xi)$  for the group  $\Gamma_\theta$  with  $\theta = \frac{21\pi}{60}$ .



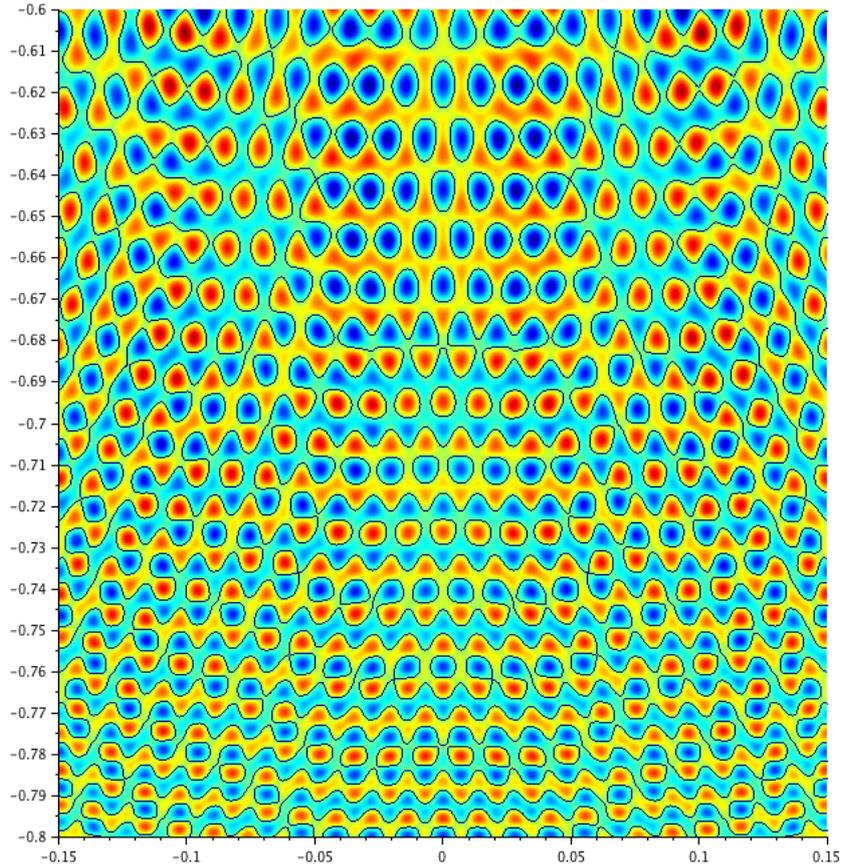
**Figure 1**

A plot of the nodal lines and level colours of  $F_\lambda(z, \xi)$  for the group  $\Gamma_\theta$  in the unit disc model. The fundamental domain is the region outside the discs whose boundaries are denoted by the blue circles. We have taken  $\theta = \frac{21\pi}{60}$ . We have set  $\xi = i$  and  $\lambda = 15$ . Blue/green corresponds to negative values while yellow/red is positive. Notice the obvious horocyclic shape of nodal lines in the vicinity of  $\xi$  which illustrates Appendix 1.



**Figure 2**

A plot of the nodal lines and level colours of  $F_\lambda(z, \xi)$  for  $\xi = i$  and  $\lambda = 50$ .



**Figure 3**

A higher frequency plot of the nodal lines of  $F_\lambda(z, \xi)$  for  $\xi = i$  and  $\lambda = 150$ . Notice the proliferation of "small" nodal domains with diameter  $\asymp \lambda^{-1}$ , which could indicate that the upper bound of Corollary 1.3 is optimal in that region.

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