THE RUELLE SPECTRUM OF GENERIC TRANSFER OPERATORS

FRÉDÉRIC NAUD

Abstract. We define a natural space of transfer operators related to holomorphic contraction systems. We show that the classical upper bounds on the Ruelle eigenvalue sequence are optimal for a dense set of transfer operators. A similar statement is derived for Perron-Frobenius operators related to uniformly expanding piecewise real analytic interval maps. The proof is based on potential theory and complex analysis.

1. Introduction

Transfer operators have a long story in the developments of the ergodic theory of hyperbolic dynamical systems, and the study of their spectrum is a key step in establishing fine statistical properties of such systems (rates of mixing, central limit theorem etc...). We refer the reader to the book [1] for a comprehensive treatment and historical references. In this paper, we will be concerned with lower bounds on the Ruelle spectrum in the case of (complexifications) of real analytic expanding maps. Let \((\phi_i)_{i \in \mathcal{I}}, \mathcal{I} = \{1, \ldots, k\}\) be a set of local inverse branches of an expanding map (see below for a more precise statement), and let \((w_i)_{i \in \mathcal{I}}\) be a finite collection of bounded holomorphic functions defined on an appropriate (non-empty) open set \(\Omega \subset \mathbb{C}^d\). Each inverse branch \(\phi_i : \Omega \to \Omega\) is assumed to be a contraction: \(\bar{\partial}(\Omega)\) is a compact subset of \(\Omega\). The Ruelle transfer operator is then defined on functions \(f\) (belonging to a suitable function space) by

\[
\mathcal{L}(f)(z) = \sum_{i \in \mathcal{I}} w_i(z)(f \circ \phi_i)(z).
\]

Ruelle [10] was the first to consider operators of this type, acting on the Banach space of bounded Holomorphic functions on \(\Omega\), proving that \(\mathcal{L}\) is a compact operator. The sequence of finite multiplicity eigenvalues (the Ruelle spectrum) \(^1\) is then denoted by \((\lambda_n(\mathcal{L}))_{n \in \mathbb{N}}\), ordered decreasingly

\[|\lambda_0| \geq |\lambda_1| \geq \ldots \geq |\lambda_n| \geq \ldots,\]

and written with multiplicities. As noticed by Bandtlow and Jenkinson in [3], there are various choices for the function space on which \(\mathcal{L}\) acts, but the Ruelle sequence \((\lambda_n(\mathcal{L}))_{n \in \mathbb{N}}\) remains the same, and depends only on the underlying expanding map and the choice of the weights \(w_i\). A convenient

1991 Mathematics Subject Classification. 37C30, 37D20.

Key words and phrases. Ruelle eigenvalues, Expanding maps, Transfer operators, Dynamical zeta functions.

\(^1\) The set of finite multiplicity eigenvalues may be finite or empty, see remarks below and the last section of this paper. In that case we set \(\lambda_n(\mathcal{L}) = 0\) for all \(n\) large (finite case) or all \(n \in \mathbb{N}\) (empty case).
choice is the Bergmann space $A^2(\Omega)$ of square integrable (with respect to Lebesgue $dV$) holomorphic functions on $\Omega$. Endowed with the natural norm
\[ \|f\|^2_{A^2(\Omega)} := \int_{\Omega} |f|^2 dV, \]
this is a Hilbert space. Acting on this space, $L$ turns out to be a compact trace class operator. The main estimate for the eigenvalues obtained by Bandtlow and Jenkinson in [2] is the following: under the above hypotheses, one can find some computable constants $^2 a, A > 0$ such that for all $n \in \mathbb{N}$,
\[ |\lambda_n(L)| \leq Ae^{-an^{1/d}}. \]
Similar estimates were previously found by various authors in different settings, see [2] for an exhaustive list of references. Except some computable cases (see below) and perturbative situations, there currently exists no general lower bound for the eigenvalue sequence, and it is in general a hard problem to prove the existence of non-trivial eigenvalues for a general transfer operator as above. One of the reasons for that is the non self-adjointness of $L$ which prevents applications of all classical techniques of spectral theory known for classical self-adjoint operators (for example the Laplacian). In this paper, we will use some techniques of potential theory to obtain lower bounds on the spectrum for a dense set of transfer operators. Let us be more precise. From now on, we assume that $\Omega$ is a bounded, non-empty and convex open subset of $\mathbb{C}^d$. By $U(\Omega)$ we denote the Banach vector space of $\mathbb{C}$-valued Holomorphic functions $f$ on $\Omega$, with a continuous extension to the closure $\overline{\Omega}$. This space is endowed with the standard supremum norm
\[ \|f\|_{U(\Omega)} := \sup_{z \in \Omega} |f(z)|. \]
By $\mathcal{K}(\Omega)$ we denote the set of holomorphic contractions of $\Omega$ i.e.
\[ \mathcal{K}(\Omega) := \{ \gamma \in U(\Omega)^d : \gamma(\overline{\Omega}) \subset \Omega \}, \]
which is an open subset of $U(\Omega)^d$. Let $k \geq 1$ be an integer. We define $\mathcal{M}_k(\Omega)$ (the set of data required to define $L$) by the product
\[ \mathcal{M}_k(\Omega) = \mathcal{K}(\Omega)^k \times U(\Omega)^k, \]
endowed with a natural product metric. Given
\[ (\phi; w) := (\phi_1, \ldots, \phi_k; w_1, \ldots, w_k) \in \mathcal{M}_k(\Omega), \]
one can define the associated transfer operator $L(\phi, w)$ defined as above, acting on the Bergmann space $A^2(\Omega)$. Remark that the topology on $\mathcal{M}_k(\Omega)$ is stronger than the operator topology on the space of bounded operators on $A^2(\Omega)$. Our main result is the following.

**Theorem 1.1.** There exists a dense subset $G$ of $\mathcal{M}_k(\Omega)$ such that for all $(\phi, w) \in G$, we have for all $\epsilon > 0$,
\[ \limsup_{n \to +\infty} \frac{|\lambda_n(L)|}{\exp(-n^{1/d+\epsilon})} = +\infty. \]
\[ ^2 \text{depending on } (\phi_i)_{i \in \mathbb{N}}, (w_i)_{i \in \mathbb{N}} \text{ and the set } \Omega \]
The formula (2) implies the following fact: for all \( \epsilon > 0 \), one can find a sequence \( (n_\epsilon(p))_{p \in \mathbb{N}} \) with \( n_\epsilon(p) \to +\infty \) as \( p \to +\infty \), such that we have for all \( p \geq 0 \),

\[
|\lambda_{n_\epsilon(p)}(L)| \geq \exp(-n_\epsilon(p)^{1/d} + \epsilon).
\]

In other words, there exists a dense set of data for which the above upper bound (1) is optimal in term of the exponent \( 1/d \). We point out that in this large class of data, there always exists \((\phi; w)\) with \( w \not\equiv 0 \) such that the spectrum of \( L_{\phi,w} \) is reduced to \( \{0\} \), see §5 for examples. It shows that the above result can only be improved in terms of the strength of genericity.

On the other hand, one may argue that the set \( M_k(\Omega) \) is a bit too large and that most transfer operators associated with \((\phi; w) \in M_k(\Omega)\) have nothing to do with the original motivation of Ruelle: transfer operators associated to real analytic expanding maps. Let us restrict for simplicity to the case of one dimensional maps, \( d = 1 \), and assume that \([0, 1] \subset \Omega\). Let us identify a relevant subset of \( M_k(\Omega) \) corresponding to marked real analytic expanding maps on \([0, 1]\).

A marking \((I, \varepsilon)\) with \( I = (I_i)_{i \in \mathbb{I}} \) and signs \( \varepsilon_i \in \{-1, +1\} \) such that
\[
\begin{align*}
\text{(1)} & \quad \cup_{i \in \mathbb{I}} I_i = [0, 1]. \\
\text{(2)} & \quad \text{For all } i \neq j, \text{Int}(I_i) \cap \text{Int}(I_j) = \emptyset.
\end{align*}
\]

The set of (inverse branches of) marked expanding maps denoted by \( E_k(I, \varepsilon)(\Omega) \) is the set of contractions \( \phi = (\phi_1, \ldots, \phi_k) \in \mathcal{K}(\Omega)^k \) satisfying the following conditions.

\[
\begin{align*}
\text{(1)} & \quad \text{For all } i \in \mathbb{I}, \text{ we have } \phi_i([0, 1]) = I_i \text{ and sup}_{x \in [0, 1]} |\phi_i'(x)| < 1. \\
\text{(2)} & \quad \text{For all } i \in \mathbb{I}, \phi_i : [0, 1] \to I_i \text{ is strictly increasing if } \varepsilon_i = +1, \text{ strictly decreasing otherwise.} \\
\text{(3)} & \quad \text{For all } i \in \mathbb{I}, \phi_i' : \Omega \to \mathbb{C} \text{ is bounded on } \Omega.
\end{align*}
\]

Thus \( E_k(I, \varepsilon)(\Omega) \) is just the set of (holomorphic extensions of) inverses branches of piecewise monotonic real analytic maps \( T \) of the unit interval, where \( T \) is simply defined by \( T|_{\text{Int}(I_i)} := \phi_i^{-1} \). Notice that by the contraction hypothesis on inverse branches, those maps \( T \) are uniformly expanding. They also satisfy a strong Markov assumption by hypothesis (1). The Perron-Frobenius operator \( \mathcal{P} \) associated to a given \( \phi \in E_k(I, \varepsilon)(\Omega) \) is a particular case of transfer operator given by

\[
\mathcal{P}(f)(x) = \sum_{i \in \mathbb{I}} \varepsilon_i \phi'_i(x)(f \circ \phi_i)(x) = \sum_{T y = x} |T'(y)|^{-1} f(y).
\]

Our second result is as follows.

**Theorem 1.2.** There exists a dense subset \( \mathcal{G} \) of \( E_k(I, \varepsilon)(\Omega) \) such that for all \((\phi, w) \in \mathcal{G}\), we have for all \( \epsilon > 0 \),

\[
(3) \quad \lim_{n \to +\infty} \sup \frac{|\lambda_n(\mathcal{P})|}{\exp(-n^{1+\epsilon})} = +\infty.
\]
Hence a dense set of expanding maps have infinitely many distinct non-trivial eigenvalues whose rate of decay is almost exponential. The strategy of the proof is to relate the "lower bounds" (2), (3) on the eigenvalue sequence to the order of certain entire functions defined as Fredholm determinants of transfer operators, which is done in §2. The problem is therefore reduced to the study of the order as a function on the "moduli space" $M_k(\Omega)$. Tools from complex analysis reviewed in §3 show that the order function is well approximated by subharmonic functions. A maximum principle imply that the order is constant except on a polar set provided that we can find at least one explicit example for which the order is maximal, which is achieved in §4 by looking at some piecewise affine contractions. In §5, we provide some non-trivial examples for which the Ruelle sequence is trivial. This complex analytic technique was previously used by Tanya Christiansen [4, 5] to obtain lower bounds for the resonance counting function in potential and euclidean scattering theory. Most of the material in this paper is inspired by [4].

2. Growth order and determinants

An efficient way to study the Ruelle spectrum via complex analysis is provided by the Fredholm determinant (Recall that $\mathcal{L}$ is trace class)

$$d(\zeta) := \det(I - \zeta \mathcal{L}) = \prod_{n \in \mathbb{N}} \left(1 - \zeta \lambda_n(\mathcal{L})\right),$$

which is an entire function on the complex plane whose zeros are obviously related to the Ruelle eigenvalues. However for our purpose, it will prove more convenient to work with the zeta function

$$Z(\zeta) := \det(I - e^{\zeta} \mathcal{L}).$$

A first important observation is the following.

**Proposition 2.1.** Assume that we have for all $n \geq 0$,

$$|\lambda_n(\mathcal{L})| \leq Ae^{-an^\rho},$$

for some constants $A,a > 0$ and $0 < \rho \leq 1$. Then one can find constants $C_1,C_2 > 0$ such that for all $\zeta \in \mathbb{C}$,

$$|Z(\zeta)| \leq C_1 e^{C_2|\zeta|^{1+1/\rho}}.$$ 

**Proof.** Recall that we have for all $\zeta \in \mathbb{C}$,

$$Z(\zeta) = \prod_{n \in \mathbb{N}} \left(1 - e^{\zeta} \lambda_n(\mathcal{L})\right),$$

the infinite product being absolutely convergent. We can write therefore for all $\zeta \in \mathbb{C}$,

$$\log |Z(\zeta)| \leq \sum_{n=0}^{+\infty} \log \left(1 + e^{\left|\zeta\right|} |\lambda_n(\mathcal{L})|\right).$$

By splitting the above sum we get ($N$ will be chosen later on)

$$\log |Z(\zeta)| \leq \sum_{n=0}^{N} \log \left(1 + e^{\left|\zeta\right|} |\lambda_0(\mathcal{L})|\right) + A e^{\left|\zeta\right|} \sum_{n=N+1}^{+\infty} e^{-an^\rho}.$$
\[ \leq C_1 + (N + 1)|\zeta| + Ae^{\left|\zeta\right|} \sum_{n=N+1}^{+\infty} e^{-\alpha n^\rho}, \]

for some constant \( C_1 > 0 \). On the other hand we have
\[ \sum_{n=N+1}^{+\infty} e^{-\alpha n^\rho} \leq \int_{N^\rho}^{+\infty} e^{-\alpha t^\rho} dt = \frac{1}{\rho} \int_{N^\rho}^{+\infty} e^{-au^1/\rho - 1} du. \]

Integrating \( \left[ \frac{1}{\rho} \right] \) times by parts shows that (\( N \) is large)
\[ \int_{N^\rho}^{+\infty} e^{-au^1/\rho - 1} du = O(N^{1-\rho}e^{-aN^\rho}). \]

We have thus obtained
\[ \log |Z(\zeta)| \leq C_1 + (N + 1)|\zeta| + Ae^{\left|\zeta\right|}N^{1-\rho}e^{-aN^\rho}. \]

Taking
\[ N = \left[ \left( \frac{2|\zeta|}{a} \right)^{1/\rho} \right] + 1 \]

yields the upper bound
\[ \log |Z(\zeta)| = O\left(|\zeta|^{1/\rho + 1}\right). \]

The proof is done. \( \Box \)

The order of the entire function \( Z(\zeta) \) is defined \(^3\) by
\[ \rho(Z) := \limsup_{r \to +\infty} \frac{\log \left( \sup_{|\zeta| \leq r} \max\{\log |Z(\zeta)|, 0\} \right)}{\log r}. \]

The upper bound (1) of Bandtlow-Jenkinson combined with the above Proposition imply that for all data \((\phi; w) \in \mathcal{M}_k(\Omega)\), \( \rho(Z) \) is at most \( d + 1 \). On the other hand, if for a given \((\phi; w) \in \mathcal{M}_k(\Omega)\), one can prove that \( \rho(Z) = d + 1 \) then again by the above Proposition, we clearly have for all \( \epsilon > 0 \),
\[ \limsup_{n \to +\infty} \frac{|\lambda_n(\zeta)|}{\exp\left(-n^{1/d+\epsilon}\right)} = +\infty, \]

otherwise \( \rho(Z) < d + 1 \). We have reduced the problem to the analysis of the order \( \rho(Z) \) over the space of data \( \mathcal{M}_k(\Omega) \).

The next step of the strategy is to restrict ourself to one dimensional families, which is done via convexity \(^4\). Indeed, for \( z \in [0, 1] \), and \( \omega \in \Omega \), set
\[ \phi(z, \omega) = (1 - z)\phi_0(\omega) + z\phi_1(\omega). \]

Clearly for all \( z \in [0, 1] \), \( \phi(z, .) \in \mathcal{K}(\Omega) \) and by compactness of \( \phi([0, 1] \times \overline{\Omega}) \subset \Omega \), one can clearly find an open connected set \( C \supset U \supset [0, 1] \) such that for

\(^3\)Here we have to write \( \max\{\log |Z(\zeta)|, 0\} \) in order to have a consistent definition for the case of constant functions \( Z(\zeta) \) with \( \log |Z(\zeta)| < 0 \). It should be noticed that whenever \( Z(\zeta) \) is non constant, the function \( r \mapsto \max_{|\zeta| \leq r} \log |Z(\zeta)| \) is a convex function of \( \log r \) and thus tends to \( +\infty \) as \( r \to +\infty \) so that the above max is no longer necessary.

\(^4\)Convexity is handy, but clearly for \( d = 1 \), simply connected is enough by conformal representation. In higher dimension, some more restrictive topological assumptions on \( \Omega \) are necessary.
all \( z \in \mathcal{U} \), \( \phi(z,.) \in \mathcal{K}(\Omega) \). For simplicity we set \( \phi(z,.) = \phi_z \). Given \( w_0, w_1 \in U(\Omega)^k \), one can define for all \( z \in \mathcal{U} \), \( w_z = zw_1 + (1-z)w_2 \in U(\Omega)^k \). The associated transfer operator \( \mathcal{L}_z := \mathcal{L}(\phi_z; w_z) \) is then defined as previously and one can check directly that it is a holomorphic operator-valued function of \( z \in \mathcal{U} \). So we have found a real analytic path in \( \mathcal{M}_k(\Omega) \) joining two arbitrary set of data \((\phi_0; w_0), (\phi_1; w_1) \in \mathcal{M}_k(\Omega)\).

In the case of Perron-Frobenius operators, i.e. if \( \phi_0, \phi_1 \in E^{I,\epsilon}_k(\Omega) \), the contraction hypothesis on inverse branches makes it possible to shrink \( \Omega \) as all sufficiently small \( \epsilon \)-neighborhood of \([0,1]\). This is due to the fact that the inverse branches are complex contracting, see [3], P. 3 for a detailed proof. Therefore the convexity hypothesis can be always satisfied in the Perron-Frobenius case. One can check readily that for all \( z \in [0,1] \), \( \phi_z \in E^{I,\epsilon}_k(\Omega) \). A simple compactness argument shows that one can find a bounded connected open set \( \mathcal{U} \supset [0,1] \) such that for all \( z \in \mathcal{U} \), \( \phi_z \in \mathcal{K}(\Omega)^k \).

Since the dependence \( z \mapsto \mathcal{L}_z \) is holomorphic on \( \mathcal{U} \), the Fredholm determinant

\[
Z(z,\zeta) := \det(I - e^{\zeta\mathcal{L}_z})
\]

is a holomorphic function on \( \mathcal{U} \times \mathbb{C} \). Both Theorems 1.1, 1.2 will follow from the following result.

**Theorem 2.2.** Using the above notations, assume that we have \( \rho(\mathcal{L}_0) = d + 1 \). Then there exists a set \( E \subset \mathcal{U} \) of Hausdorff dimension 0, such that for all \( z \in \mathcal{U} \setminus E \), \( \rho(Z(z)) = d + 1 \).

The above statement shows that in one dimensional families, the upper bound is optimal for a generic set of data in a strong metric sense. The complementary of a set of zero dimension being dense, Theorem 1.1 holds if one can find at least one example \((\phi_0; w_0) \in \mathcal{M}_k(\Omega)\) for which \( \rho(Z(z)) = d + 1 \). Similarly, \([0,1] \cap E \) has zero Hausdorff dimension, hence its complementary in \([0,1]\) is dense. Therefore Theorem 1.2 is true if one can find an example with maximal order in each space \( E^{I,\epsilon}_k(\Omega) \). This task is postponed to §4 while the next section is focused on Theorem 2.2.

### 3. Plurisubharmonic functions and growth order

In this section we collect some basic definitions and recall, mostly without proofs, the material necessary to prove Theorem 2.2 which is the central tool of this paper. Our references are [8] for the theory of several complex variables and [9] for potential theory in the one dimensional case. In our applications, we mainly need to look at the case of two and one complex variables, but we state the results in the \( n \)-dimensional case. Let \( \mathcal{O} \subset \mathbb{C}^n \) be an open connected non-empty set. We denote by \( D(a,r) \subset \mathbb{C} \) the closed euclidean disc centered at \( a \) and of radius \( r \).

**Definition 3.1.** A real valued function \( \varphi : \mathcal{O} \to [-\infty, +\infty) \) is said to be plurisubharmonic on \( \mathcal{O} \) if

---

5One can check directly that \( z \mapsto \mathcal{L}_z \) is holomorphic as a function from \( \mathcal{U} \) to \( B(A^2(\Omega)) \), the space of bounded operators on \( A^2(\Omega) \). Analyticity of the determinant then follows from the so-called Plemelj-Smithies expansions, see for example [7].
(1) \( \varphi \) is upper semi-continuous and \( \varphi \not\equiv -\infty \) on \( \emptyset \).
(2) For all \( z \in \emptyset \), for all \( r > 0 \) and \( w \in \mathbb{C}^n \) such that \( z + wD(0,r) \subset \emptyset \),
\[
\varphi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + we^{i\theta}) d\theta.
\]

We denote by \( \text{PSH}(\emptyset) \) the set of plurisubharmonic functions on the domain \( \emptyset \). From the above definition one derives the following basic properties.

**Proposition 3.2.** Using the above notations, we have the following.

1. \( \text{PSH}(\emptyset) \) is stable under positive linear combinations.
2. If \( \varphi_1, \varphi_2 \in \text{PSH}(\emptyset) \), then \( \max\{\varphi_1, \varphi_2\} \in \text{PSH}(\emptyset) \).
3. \( \text{PSH}(\emptyset) \subset L^1_{\text{loc}}(\emptyset) \).
4. If \( f : \emptyset \to \mathbb{C} \) is a non identically zero holomorphic function, then \( \varphi(z) = \log |f(z)| \) is plurisubharmonic.

A subset \( E \subset \emptyset \) is said to be pluripolar if there exists a subharmonic function \( \varphi : \emptyset \to [-\infty, +\infty) \) such that \( E \subset \{z \in \emptyset : \varphi(z) = -\infty\} \). From property (3) of the above Proposition it follows that every pluripolar set is measurable with zero \( 2n \)-dimensional Lebesgue measure. In the one dimensional case \( n = 1 \), one can show, [9] P. 57, that every Borel\(^6\) polar set has zero Hausdorff dimension. It can be however uncountable, see [9] P.143, for examples of Cantor-like polar sets. One of the key features of plurisubharmonic functions is the following.

**Proposition 3.3.** (Maximum principle)
Given \( \varphi \in \text{PSH}(\emptyset) \), either we have for all \( z \in \emptyset \),
\[
\varphi(z) < \sup_{x \in \emptyset} \varphi(x),
\]
or \( \varphi = \sup_{\emptyset} \varphi \) is a constant.

Let \( \mathcal{U} \subset \mathbb{C} \) be a domain and let \( \varphi = \varphi(z, w) : \mathcal{U} \times \mathbb{C} \to [-\infty, +\infty) \) be a plurisubharmonic function. For all \( z \in \mathcal{U} \) we define the order of growth \( \rho(z) \) (with respect to \( w \)) by
\[
\rho_{\varphi}(z) := \limsup_{r \to +\infty} \frac{\log(\sup_{|w| \leq r} \max\{\varphi(z, w), 0\})}{\log r}.
\]

In general, \( z \mapsto \rho_{\varphi}(z) \) is not a subharmonic function so the above maximum principle cannot be applied, however we have the following key result ([8], P.25).

**Proposition 3.4.** Assume that \( \varphi \in \text{PSH}(\mathcal{U} \times \mathbb{C}) \) and that \( \varphi \geq 1 \). Then for all relatively compact domains \( \mathcal{U}' \subset \mathcal{U} \), there exists a sequence of negative functions \( \psi_q \in \text{PSH}(\mathcal{U}') \) such that for all \( z \in \mathcal{U}' \),
\[
-\frac{1}{\rho_{\varphi}(z)} = \limsup_{q \to +\infty} \psi_q(z).
\]

To prove Theorem 2.2, we need in addition the following fact ([8], P.25) which replaces the maximum principle.

\(^6\)Non Borel pluripolar sets do exists, though they are Lebesgue measurable. However for our applications we will always encounter Borel sets.
Proposition 3.5. Let $(\varphi_q)$ be a sequence in $\text{PSH}(\mathcal{O})$, uniformly bounded from above on compact subsets. Assume that $\limsup_{q \to +\infty} \varphi_q \leq 0$ and that there exists $\xi \in \mathcal{O}$ such that $\limsup_{q \to +\infty} \varphi_q(\xi) = 0$. Then
\[
\limsup_{q \to +\infty} \varphi_q = 0,
\]
except on a Borel pluripolar subset of $\mathcal{O}$.

Let us show how to deduce Theorem 2.2. We use the notations of §2. Consider the plurisubharmonic function on $U \times \mathbb{C}$ given by
\[
\varphi(z, \zeta) = \max\{\log |Z(z, \zeta)|, 1\} \geq 1.
\]
We know that for all $z \in U$, $\rho_\varphi(z) \leq d + 1$. Let $U' \subset U$ be relatively compact in $U$ such that $0 \in U'$ and use Proposition 3.4 to find a sequence of negative subharmonic functions $\psi_q$ on $U'$ such that
\[
-1/\rho_\varphi(z) = \limsup_{q \to +\infty} \psi_q(z).
\]
For all $q \in \mathbb{N}$, set $\varphi_q(z) = \psi_q(z) + 1/(d + 1)$. Clearly each $\varphi_q$ is subharmonic on $U'$, the sequence is uniformly bounded and $\limsup_{q \to +\infty} \varphi_q \leq 0$. On the other hand, we know that
\[
\rho(Z(0)) := \limsup_{r \to +\infty} \frac{\log \left( \sup_{|\zeta| \leq r} \max\{\log |Z(0, \zeta)|, 0\} \right)}{\log r} = d + 1.
\]
Now $\zeta \mapsto Z(0, \zeta)$ is a non constant entire function hence
\[
M(r) = \sup_{|\zeta| \leq r} \log |Z(0, \zeta)|
\]
tends to $+\infty$ as $r \to +\infty$, therefore we have $\rho(Z(0)) = \rho_\varphi(0) = d + 1$. Applying Proposition 3.5 (with $\xi = 0$), we obtain that $\rho_\varphi(z) = d + 1$ for all $z \in U' \setminus E$ where $E$ is a Borel polar subset of $U'$. Using a countable exhaustion of $U$ by relatively compact subsets and the fact that a countable union of polar sets is polar (see [8], P.24), we get that $\rho_\varphi(z) = d + 1$ except on a Borel polar subset of $U$, hence of Hausdorff dimension 0. The proof ends by remarking as above that if $\rho_\varphi(z) = d + 1$, then we have actually $\rho(Z(z)) = \rho_\varphi(z) = d + 1$.

4. Affine contractions and a determinant

In this section, we provide examples of transfer operator $L_f$ for which the order of the zeta function $Z(\zeta)$ is computable and conclude by the proof of Theorem 1.1 and Theorem 1.2. We point out that a similar computation of the spectrum, for one affine contraction, was done by Fried in [6]. Choose some factors $r_i \in \mathbb{R}$, $i \in \mathcal{I}$ with $0 < |r_i| < 1$. Set for all $i \in \mathcal{I}$, $\varepsilon_i = \text{sign}(r_i)$. Choose some $q_i \in \mathbb{C}^d$, $i \in \mathcal{I}$, and assume that each map $\gamma_i(z) := r_i z + q_i$ belongs to $\mathcal{K}(\Omega)$ for a suitable $\Omega$. Now we consider the transfer operator
\[
L(f)(z) = \sum_{i \in \mathcal{I}} \varepsilon_i r_i (f \circ \gamma_i)(z),
\]
where $f \in A^2(\Omega)$. 
Proposition 4.1. With the above choice of $\mathcal{L}$, the Fredholm determinant $Z(\zeta) = \det(I - e^{\zeta} \mathcal{L})$ is an entire function of order $\rho(Z) = d + 1$.

Proof. For all $\text{Re}(\zeta) < 0$, we have the absolutely convergent expansion

$$Z(\zeta) = \exp \left( - \sum_{n=1}^{+\infty} \frac{e^{n\zeta}}{n} \text{Tr}(\mathcal{L}^n) \right).$$

The trace of the $n$th iterate of $\mathcal{L}$ is classically given by (see for example [2])

$$\text{Tr}(\mathcal{L}^n) = \sum_{\alpha \in \mathcal{P}^n} e^{r_{\alpha}^n} \varepsilon_{\alpha} r_{\alpha}^n,$$

where $r_{\alpha} = r_{\alpha_1} \ldots r_{\alpha_n}$, $\varepsilon_{\alpha} = \varepsilon_{\alpha_1} \ldots \varepsilon_{\alpha_n}$, and $z_{\alpha}$ is the unique fixed point of $\gamma_{\alpha} = \gamma_{\alpha_1} \circ \ldots \circ \gamma_{\alpha_n} : \Omega \to \Omega$.

In our particular case the computation is easy and we get

$$\det(I - D\gamma_{\alpha}(z_{\alpha})) = (1 - r_{\alpha})^d.$$

Hence for all $\text{Re}(\zeta) < 0$ we can write

$$Z(\zeta) = \exp \left( - \sum_{\ell=0}^{+\infty} \frac{d(d+1) \ldots (d+\ell-1)}{\ell!} \sum_{n=1}^{+\infty} \frac{e^{n\zeta}}{n} \sum_{\alpha \in \mathcal{P}^n} e^{r_{\alpha}^\ell} \varepsilon_{\alpha} r_{\alpha}^\ell \right).$$

Now consider the $k \times k$ matrix

$$A_{\ell} = \begin{pmatrix} e_{11}^{\ell+1} & e_{21}^{\ell+1} & \ldots & e_{k1}^{\ell+1} \\ e_{12}^{\ell+1} & e_{22}^{\ell+1} & \ldots & e_{k2}^{\ell+1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1k}^{\ell+1} & e_{2k}^{\ell+1} & \ldots & e_{kk}^{\ell+1} \end{pmatrix},$$

then a simple induction shows that

$$\text{Tr}(A_{\ell}^n) = \sum_{\alpha \in \mathcal{P}^n} e_{\alpha} r_{\alpha}^{\ell+1},$$

so that

$$Z(\zeta) = \prod_{\ell=0}^{+\infty} \left( \det(I - e^{\zeta} A_{\ell}) \right)^{m_{\ell}} = \prod_{\ell=0}^{+\infty} \left( 1 - e^{\zeta} \sum_{i \in I} e^{r_{i}^{\ell+1}} \right)^{m_{\ell}},$$

where we have set

$$m_{\ell} = \frac{d(d+1) \ldots (d+\ell-1)}{\ell!}.$$

Because the above infinite product is convergent for all $\zeta \in \mathbb{C}$, this equality actually holds for all $\zeta \in \mathbb{C}$ by analytic continuation. Denote by $\mathcal{Z}$ the set of zeros of $Z(\zeta)$ and define the counting function

$$N(R) = \# \{ w \in \mathcal{Z} : |w| \leq R \}.$$
each \( w_{p,q} \) counted with multiplicity \( m_{2p} \). Set \( r^* = \max_{i \in \mathbb{J}} |r_i| \), and let \( m^* \) denote the number of occurrences of \( r^* \) in \( \{|r_i|, \ i \in \mathbb{J} \} \). As \( p \to +\infty \) we have
\[
w_{p,q} = (2p + 1)|\log r^*| + \log m^* + 2iq\pi + o(1).
\]
This shows that for \( R \) large, we have
\[
N(R) \geq CR \sum_{2p \leq R} m_{2p},
\]
where \( C > 0 \) is a suitable constant. It is an exercise to check that we have
\[
m_i \geq B\ell^{d-1},
\]
for some \( B > 0 \), which implies that for large \( R \),
\[
N(R) \geq \tilde{C}R^{d+1},
\]
for some constant \( \tilde{C} > 0 \). Assume that the growth order
\[
\rho(Z) = \limsup_{r \to +\infty} \frac{\log \left( \sup_{|\zeta| = r} \log |Z(\zeta)| \right)}{\log r} < d + 1.
\]
Then for all \( \epsilon > 0 \), one can find some constants \( A > 0 \) such that for all \( R \) large,
\[
\sup_{|\zeta| = R} \log |Z(\zeta)| \leq AR^{\rho(Z)+\epsilon}.
\]
Now by Jensen’s formula applied to \( Z(\zeta) \) on the disc \( D(0,2R) \), we get
\[
(\log 2)N(R) + O(\log R) \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log |Z(2Re^{i\theta})|d\theta,
\]
which implies that
\[
N(R) = O \left( R^{\rho(Z)+\epsilon} \right),
\]
a contradiction with the above lower bound if \( \epsilon \) is small enough. \( \square \)

We are now in position to complete the proof of Theorem 1.1 and 1.2. Assume for simplicity that \( 0 \in \Omega \), and choose \( y_1, \ldots, y_k \in \Omega \) together with a set of \( \delta_i > 0, \ i \in \mathbb{J} \) such that each \( B(y_i, \delta_i) \subset \Omega \). Now for all \( i \in \mathbb{J} \), boundedness of \( \Omega \) implies that there exist \( 1 > r_i > 0 \) so that \( r_i\Omega \subset B(0, \delta_i) \).

We look at the set of contractions defined by \( \phi_{i,0}(z) = r_i z + y_i \). Obviously each \( \phi_{i,0} \in \mathcal{K}(\Omega) \), and choose the weights \( w_{i,0} = r_i \). We now set \( \phi_0 = (\phi_{1,0}, \ldots, \phi_{k,0}) \) and \( w_0 = (w_{1,0}, \ldots, w_{k,0}) \). By the above Proposition, the order of the entire function \( Z(0,\zeta) := \det(I - e^{\zeta L_{\phi_0,w_0}}) \) is exactly \( d + 1 \). Consider now an arbitrary \( (\phi_1, w_1) \in \mathcal{M}_k(\Omega) \). Using the notations of §2 and applying Theorem 2.2, there exists an open connected set \( U \supset [0,1] \) and a set of zero Hausdorff dimension \( E \subset \mathbb{U} \) such that for all \( z \in \mathbb{U} \setminus E \), the order of \( Z(z,\zeta) \) is exactly \( d + 1 \). The set \( E \) being of zero two-dimensional Lebesgue measure, for all \( \eta > 0 \) small enough, one can find \( z \in B(1, \eta) \) with \( z \notin E \). This allows to find elements \( (\phi, w) \in \mathcal{M}_k(\Omega) \) that are arbitrarily close to \( (\phi_1, w_1) \) for which the order of the determinant \( Z_{\phi,w}(\zeta) \) remains equal to \( d + 1 \). The proof of Theorem 1.2 follows the same lines except that we work in \( \mathcal{L}_k \) and we remark that the one dimensional Lebesgue measure of \( E \cap [0,1] \) is zero which allows to conclude.
5. Examples with no point spectrum

In this short section, we show how to build non-trivial examples \((\phi, w) \in \mathcal{M}_k(\Omega)\) for which the spectrum of \(L\) is simply \(\{0\}\). Assume for simplicity that \(0 \in \Omega\), and set

\[ V = \{(x_1, \ldots, x_{d-1}, 0) \in \Omega : x_i \in \mathbb{C}\}. \]

Pick 7 \(y_1, \ldots, y_k \in V\) and let \(\delta_i > 0\) be such that \(B(y_i, \delta_i) \subset \Omega\) for all \(i \in \mathcal{I}\). Because \(\Omega\) is bounded, we can choose for all \(i \in \mathcal{I}\) a constant \(r_i \in \mathbb{R}\) such that \(r_i \Omega \subset B(0, \delta_i)\). Now for all \(i \in \mathcal{I}\), we set

\[ \phi_i(z) := r_i z + y_i. \]

Clearly each \(\phi_i \in \mathcal{X}(\Omega)\) and \(\phi_i(V) \subset V\). There exists non trivial holomorphic functions \(w_i \in U(\Omega)\) such that \(w_i(z) = 0\) if and only if \(z \in V\). Computing the determinant \(Z(\zeta)\) associated to such a \((\phi, w)\) for small \(|\zeta|\) shows that

\[ Z(\zeta) = \exp \left( -\sum_{n=1}^{\infty} \frac{e^{n\zeta}}{n} \sum_{\alpha \in \mathbb{N}} \det(I - D\gamma_\alpha(z_\alpha)) \right), \]

where \(z_\alpha \in V\) is the unique fixed point of

\[ \gamma_\alpha = \gamma_{\alpha_1} \circ \cdots \gamma_{\alpha_n} : \Omega \to \Omega, \]

and \(w_\alpha\) is given by

\[ w_\alpha = w_{\alpha_1}(\phi_{\alpha_2} \cdots \phi_{\alpha_n} z_\alpha)w_{\alpha_2}(\phi_{\alpha_3} \cdots \phi_{\alpha_n} z_\alpha) \cdots w_{\alpha_n}(z_\alpha). \]

Obviously, \(w_\alpha = 0\) for all \(\alpha\) and we can conclude by analytic continuation that \(Z(\zeta) = 1\) for all \(\zeta \in \mathbb{C}\) which in view of the formula

\[ Z(\zeta) = \prod_{n \in \mathbb{N}} (1 - e^{\zeta \lambda_n}) \]

implies that for all \(n \geq 0\), \(\lambda_n = 0\) and thus the Ruelle spectrum is trivial. More sophisticated examples can be produced by looking at a general \((d - 1)\) dimensional analytic submanifold \(V \subset \Omega\) preserved by a set of contractions. Transfer operators with weights vanishing on \(V\) have an empty point spectrum.

Acknowledgements. Part of this material was thought of while attending the conference ”Dynamical systems in Denton 2” which took place at the university of North Texas. I wish to thank to the organizers for financial support and accommodations. I would also like to thank Hans H. Rugh for pointing out examples of transfer operators with empty point spectrum and the referee for his comments.

References


7If \(d = 1\) then \(V\) is just \(\{0\}\). The points \(y_1, \ldots, y_k\) can be distincts if \(d \geq 2\).


Laboratoire d’Analyse non-linéaire et Géométrie, Université d’Avignon, 33 rue Louis Pasteur, 84000 Avignon France.

E-mail address: frederic.naud@univ-avignon.fr